4

Relativistic Fluid Flow

In many radiation hydrodynamics problems of astrophysical interest, the fluid moves at extremely high velocities, and relativistic effects become important. Examples of such flows are supernova explosions, the cosmic expansion, and solar flares. To account for relativistic effects on a macroscopic level, it is usually adequate to adopt a continuum view, without inquiring in detail into the nature of the fluid itself; such an approach is obviously appropriate for a high-velocity flow of moderate-temperature, low-density gas. In some cases, however, the fluid exhibits relativistic effects on a microscopic level. These situations require a kinetic theory approach, which, in addition, has the advantage of providing precise definitions of, and relations among, the thermodynamic properties of the material.

In what follows we shall develop both the continuum and kinetic theory views of the dynamics of relativistic ideal fluids, thereby retaining parallelism with our earlier work in the nonrelativistic limit, while at the same time laying a thorough groundwork for the treatment of radiation in Chapters 6 and 7. For relativistic nonideal fluids, we consider the continuum view only, obtaining covariant generalizations of the results in Chapter 3.

The flows that are of primary importance to us in this book can be treated entirely within the framework of special relativity. Nevertheless, many of the equations derived in this chapter are completely covariant and apply in general relativity. Only in §§95 and 96 will we need to forsake inertial frames and work in a Riemannian spacetime. Excellent accounts of the theory of general relativistic flows are given in (**L4**); Chapters 5, 22, and 26 of (**M3**); (**T1**); and Chapter 11 of (**W2**). Numerical methods for solving general-relativistic flow problems are discussed in (**M1**), (**M2**), and (**W3**).

4.1 Basic Concepts of Special Relativity

In this section we summarize the ideas from special relativity needed to obtain the equations of hydrodynamics in covariant form. For a more complete discussion the reader can consult the many texts available [e.g., (**A1**), (**A2**), (**B1**), (**L1**), (**L2**), (**L3**), (**M4**), (**R1**), (**S1**), and (**W2**)].

34. The Relativity Principle

In Newtonian mechanics, one presupposes the existence of an *absolute space* of three dimensions in which one can choose rigid *reference frames*; furthermore, one assumes that time is a *universal* independent variable. Among all such frames the preferred frames are the nonaccelerating *inertial frames.* From an operational point of view one attempts to define a reference frame fixed in absolute space through observations of extremely distant objects (e.g., distant galaxies and quasars). We thus obtain a *fundamental reference system* that (1) does not rotate with respect to the large-scale distribution of matter in the Universe and (2) is symmetrical in the sense that when we account for the expansion of the Universe, all material appears to recede isotropically from the observer.

All frames moving uniformly with respect to this fundamental system are inertial frames. Consider a system S' moving uniformly with velocity $\mathbf{v}$ relative to a system S. Accepting the Newtonian view of space and time, one would transform coordinates between S and S' by means of a *Galilean transformation*:

$$x' = x - v_x t, \tag{34.1a}$$

$$y' = y - v_y t, \tag{34.1b}$$

$$z' = z - v_z t, \tag{34.1c}$$

and

$$t' = t. \tag{34.1d}$$

Because S' does not accelerate with respect to S, an isolated body moving with constant velocity $\mathbf{v}_0$ in S will appear to move with a (different) constant velocity $\mathbf{v}_0'$ in S'. Furthermore, because (34.1) implies that $\ddot{\mathbf{x}}' = (d^2\mathbf{x}'/dt^2) \equiv (d^2\mathbf{x}/dt^2) = \ddot{\mathbf{x}}$, if we assume with Newton that a mechanical force $\mathbf{f}$ is the same in all inertial frames, then we conclude that $m\ddot{\mathbf{x}} = \mathbf{f} = m\ddot{\mathbf{x}}'$. Thus Newton's laws of motion are *invariant* under a Galilean transformation. Therefore the dynamical behavior of all mechanical systems governed by Newton's laws is identical in all inertial frames; that is, in Newtonian mechanics all inertial frames are dynamically equivalent, a property referred to as *Newtonian* (or *Galilean*) *relativity.*

With the development of Maxwell's theory of electromagnetism, it was realized that light travels with a unique speed c in vacuum; it was hypothesized that this propagation occurs in a "luminiferous ether," which was assumed to be at rest relative to Newtonian absolute space. A consequence of applying the Galilean transformation to Maxwell's equations is that the velocity of light measured by an observer should depend on his motion relative to the ether. In particular, because laboratories on Earth share its distinctive motion through absolute space, one should be able to detect the drift of the ether past the lab. As is well known, sensitive experiments reveal no such effect, and one is forced to conclude that the Newtonian concepts of space and time are faulty.

The difficulties just described were completely overcome by Einstein's theory of *special relativity*, which is based on two fundamental principles.

First, Einstein asserted his *relativity principle*, which states that all inertial frames are completely equivalent for performing all physical experiments. This is a sweeping generalization, because it implies that *all* laws of physics, not just Newton's laws, must have an invariant form (i.e., must be *covariant*) when we change from one inertial frame to another. In particular, a correct formulation of a physical law must not contain any reference, explicit or implicit, to the velocity, relative to some "absolute space", of the coordinate system in which the phenomenon is described. The demand for covariance of valid physical laws immediately suggests that one must attempt to formulate them as *tensor equations*, which, as we already know, have precisely this property.

Second, Einstein asserted (in agreement with experiment but contrary to "common sense") that the speed of light is the same in all inertial frames, independent of the motion of the source. This postulate has profound implications because it is incompatible with Galilean transformations, in which space and time are independent (cf. §35). Instead, a new transformation, the *Lorentz transformation*, is required, which couples space and time into a single entity, *spacetime*, defined in such a way that Einstein's second postulate is satisfied. We then choose as valid physical laws those formulations that are covariant under Lorentz transformation.

35. The Lorentz Transformation

THE SPECIAL LORENTZ TRANSFORMATION

Let system S' move with uniform speed v along the z axis of system S, and at the instant the origins of S and S' coincide, set $t = t' = 0$. At that instant, suppose that a light pulse is emitted from the origin. According to the two postulates of special relativity, observers in both frames must observe a spherical wavefront, centered on the origin of their system, propagating with velocity c. That is, an observer in S will describe the wavefront by the equation

$$x^2 + y^2 + z^2 - c^2t^2 = 0, \qquad (35.1)$$

and an observer in S' by the equation

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0. \qquad (35.2)$$

Direct substitution from (34.1) shows that with the Galilean transformation we cannot satisfy both of these equations simultaneously.

We seek to modify the transformation law so that it guarantees compatibility of (35.1) and (35.2), while reducing to the Galilean transformation in the limit of very low velocities, where our everyday experience applies. For $v_x = v_y = 0$, the terms in x and y offer no trouble, but we must derive new transformations for z and t. If we assume that the transformation preserves the *homogeneity* of space, so that all points in space and time have equivalent transformation properties, then we conclude that the transformation equations must be *linear*. Thus we hypothesize a transformation of the form

$$z' = \gamma(z - vt), \tag{35.3}$$

where γ is a constant to be determined, which reduces to unity for vanishingly small velocities. We must also modify (34.1d) because no transformation of space coordinates alone can yield wavefronts that are simultaneously spheres in both systems. We thus try the linear transformation

$$t' = At + Bz \tag{35.4}$$

where A and B must also be determined.

Substituting (35.3) and (35.4) into (35.2) and comparing with (35.1) we find

$$\gamma^2 - B^2c^2 = 1, \tag{35.5}$$

$$A^2c^2 - \gamma^2 v^2 = c^2, \tag{35.6}$$

and

$$ABc^2 + \gamma^2 v = 0. \tag{35.7}$$

From these three equations we find that A, B, and γ are

$$\gamma = A = (1 - v^2/c^2)^{-1/2} \tag{35.8}$$

and

$$B = -\gamma v/c^2. \tag{35.9}$$

Using the customary notation $\beta \equiv v/c$, the Lorentz transformation equations are

$$x' = x, \tag{35.10a}$$

$$y' = y, \tag{35.10b}$$

$$z' = \gamma(z - vt), \tag{35.10c}$$

and

$$t' = \gamma(t - \beta z/c), \tag{35.10d}$$

For $v/c \ll 1$, (35.10a) to (35.10c) reduce to the Galilean transformation; however, the transformation from t to t' still depends on v to first order, a fact ignored in the Galilean transformation through the assumption of the existence of an absolute universal time.

Thus the Lorentz transformation turns out to be a *four-dimensional* transformation in spacetime. In spacetime, let us now choose coordinates $x^{(0)} \equiv ct$, $x^{(1)} \equiv x$, $x^{(2)} \equiv y$, and $x^{(3)} \equiv z$. Then if $\mathbf{x}$ is a four-component column vector, (35.10) can be written in matrix notation as

$$\mathbf{x}' = \mathbf{L}\mathbf{x} \tag{35.11}$$

where $\mathbf{L}$ is the matrix

$$\mathbf{L}=\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{35.12}$$

In component form, (35.11) can be written

$$x'^{\alpha} = L^{\alpha'}_{\beta} x^{\beta} \tag{35.13}$$

where $L^{\alpha'}_{\beta}$ is the element in the α'th row and βth column of $\mathbf{L}$; from (35.13) it is obvious that

$$(\partial x^{\alpha'}/\partial x^{\beta}) = L^{\alpha'}_{\beta}. \tag{35.14}$$

Notice that $\mathbf{L}$ is symmetric, so that $\mathbf{L}^t \equiv \mathbf{L}$. Furthermore, it is easy to show by direct calculation that $|\mathbf{L}| = 1$. We demand that all valid Lorentz transformations have unit determinant; the significance of this requirement will become clearer below when we discuss Minkowski coordinates.

The matrix $\mathbf{L}$ transforms quantities from S to S'; clearly there must be an *inverse transformation* $\mathbf{L}^{-1}$ such that

$$\mathbf{x} = \mathbf{L}^{-1}\mathbf{x}'. \tag{35.15}$$

By direct calculation of the inverse of $\mathbf{L}$ one readily finds

$$\mathbf{L}^{-1}=\begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{35.16}$$

This result is precisely what we would expect on physical grounds because if observer O in S sees S' moving with velocity v along the z axis, then from the principle of relativity O′ in S' must see S moving with velocity $-v$ along his z' axis. Both points of view are equally valid, hence the transformation from S' to S must be the same as that from S to S', but with the sign of v reversed; equation (35.16) has precisely this property. In component form

$$x^{\alpha} = L^{\alpha}_{\beta'} x^{\beta'}, \tag{35.17}$$

where $L^{\alpha}_{\beta'}$ is the element in the αth row and β'th column of $\mathbf{L}^{-1}$. Hence

$$(\partial x^{\alpha}/\partial x'^{\beta}) = L^{\alpha}_{\beta'}. \tag{35.18}$$

Finally, note in passing that the inverse relationship between $\mathbf{L}$ and $\mathbf{L}^{-1}$ implies that

$$L^{\alpha}_{\gamma'} L^{\gamma'}_{\beta} = \delta^{\alpha}_{\beta} \tag{35.19a}$$

and

$$L^{\alpha'}_{\gamma} L^{\gamma}_{\beta'} = \delta^{\alpha'}_{\beta'}. \tag{35.19b}$$

LORENTZ-FITZGERALD CONTRACTION AND TIME DILATION

Let us now examine some physical consequences of the Lorentz transformation. One is that a standard measuring rod will be found to have a different length by observers in the two frames S and S'. From (35.10) we see that $\Delta x' = \Delta x$ and $\Delta y' = \Delta y$, hence the lengths of intervals perpendicular to $\mathbf{v}$ are the same in both frames. But suppose that we have a rod of length Δz at rest along the z axis in S, and an observer in S' measures its length at a given instant t'. Then, using $z = \gamma(z' + \beta ct')$ from (35.15), we see that

$$\Delta z' = \Delta z/\gamma; \tag{35.20}$$

that is, the rod appears to have contracted by a factor $(1 - v^2/c^2)^{1/2}$ when it moves relative to the observer making the measurement. This is known as the *Lorentz–Fitzgerald contraction effect.* The length of a rod is greatest when it is at rest relative to an observer; this is its *proper length.*

Similarly, suppose we have a clock at rest in S which is observed by O′ in S'; then from (35.10d), we find

$$\Delta t' = \gamma\, \Delta t; \tag{35.21}$$

that is, the clock appears to run more slowly when it is in motion relative to the observer. This is the *time-dilation effect.* When a clock is at rest relative to an observer it keeps *proper time,* and appears to go at its fastest rate.

Equations (35.20) and (35.21) imply that the *spacetime volume element* is invariant, that is,

$$dV\, dt = dx\, dy\, dz\, dt \equiv dx'\, dy'\, dz'\, dt' = dV'\, dt', \tag{35.22}$$

where dV and dV' denote ordinary three-dimensional volume elements in S and S'. We shall use this result repeatedly.

THE SPACETIME INTERVAL

An *event* (or *world point*) in spacetime is specified by its four coordinates $(x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}) = (ct, x, y, z)$ which tell when and where the event occurs. The sequence of world points belonging to a real particle is called its *world line,* which describes the particle's motion in spacetime. The *spacetime interval ds* between two events is defined to be

$$ds^2 = -c^2\, dt^2 + dx^2 + dy^2 + dz^2 \equiv -c^2\, d\tau^2; \tag{35.23}$$

(35.23) also defines the interval of *proper time,* $d\tau$.

The arrangement of signs $(-;\ +,\ +,\ +)$ in (35.23) is known as the *signature* of the metric; the choice made here is the "spacelike" convention. Some authors use the "timelike" convention $(+; -, -, -)$. Both choices are equally valid, but formulae using the two conventions can differ in the signs given to various terms; it is essential to check the signatures of the metrics when comparing formulae from different sources.

The importance of the expression (35.23) for the spacetime interval is that it is a *world scalar*, that is, it is invariant under Lorentz transformation, and hence provides a coordinate-independent measure of the "distance" between events in spacetime. The invariance of both the form and the value of ds^2 under the special Lorentz transformation derived above can be easily verified by direct substitution from (35.10). We will demand that general Lorentz transformations (derived below), which describe the motion of S' relative to S along an arbitrary direction, must also preserve invariance of the spacetime interval; this requirement helps to determine the form of the general transformation laws.

We can write the interval, which is the *line element* of spacetime, in standard metric form (cf. §A3.4) as

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta, \tag{35.24}$$

where Greek indices run from 0 to 3. In this tensor form it is obvious that the interval has an invariant value under coordinate transformation. The tensor

$$\boldsymbol{\eta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{35.25}$$

is called the *Lorentz metric*. While the metric in ordinary three-dimensional space is positive definite (being the sum of squares), the Lorentz metric is *indefinite*, and ds^2 may be positive, negative, or zero.

Transformations of coordinate systems imply transformations of the metric tensor according to the standard rule [cf. equation (A3.17)]

$$\eta'_{\alpha\beta} = \frac{\partial x^\varepsilon}{\partial x'^\alpha}\frac{\partial x^\zeta}{\partial x'^\beta}\,\eta_{\varepsilon\zeta} = L^\varepsilon_{\alpha'} L^\zeta_{\beta'} \eta_{\varepsilon\zeta} \tag{35.26}$$

where we have used (35.18). In view of the invariance of the *form* of ds^2 under Lorentz transformation we know that $\eta'_{\alpha\beta} = \eta_{\alpha\beta}$; hence we accept as valid Lorentz transformations only those transformations for which

$$L^\varepsilon_{\alpha'} L^\zeta_{\beta'} \eta_{\varepsilon\zeta} \equiv \eta_{\alpha\beta}. \tag{35.27}$$

It is easily verified that (35.27) is satisfied by the special Lorentz transformation derived above. Finally, it is worth noting that the reciprocal tensor satisfies $\eta^{\alpha\beta} \equiv \eta_{\alpha\beta}$, and that $\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^\gamma_\alpha$.

Intervals can be classified into three categories: they are called *spacelike* if $ds^2 > 0$, *timelike* if $ds^2 < 0$, and *null* if $ds^2 = 0$. Because the interval is invariant this categorization is unique, and an interval which is, say, spacelike in one frame will be spacelike in all frames. For two events separated by a spacelike interval there always exists a Lorentz transformation to a particular frame in which the two events occur simultaneously at two different locations. Similarly, for two events separated by a timelike interval there always exists a Lorentz transformation such that in some particular frame the two events occur at the same location at two successive times. From this fact one sees that the world lines of physical particles must be timelike.

Null intervals describe photon paths. From any given event in spacetime the ensemble of all photon paths generates a *null cone* along which light signals propagate. The volume around the x^0 axis contained within the cone comprises all timelike intervals, and can be separated into an absolute future and an absolute past relative to the event at the origin. The volume of spacetime exterior to the null cone comprises all spacelike intervals and represents a conditional "present" in which events are absolutely separated in space.

Finally, let us reconsider the invariance of the spacetime volume element in the light of the metric form for the interval. We know from general considerations [cf. equation (A3.21)] that

$$\sqrt{-\eta}\, dx^{(0)}\, dx^{(1)}\, dx^{(2)}\, dx^{(3)} = \sqrt{-\eta'}\, dx'^{(0)}\, dx'^{(1)}\, dx'^{(2)}\, dx'^{(3)}, \quad (35.28)$$

where $\eta \equiv |\eta_{\alpha\beta}|$ and $\eta' \equiv |\eta'_{\alpha\beta}|$; minus signs appear in (35.28) because the determinants are negative. But $\eta'_{\alpha\beta} \equiv \eta_{\alpha\beta}$, hence $\eta' \equiv \eta$. Thus (35.28) immediately leads back to (35.22).

THE GENERAL LORENTZ TRANSFORMATION

The special Lorentz transformation derived above applies when S' moves along the z axis of S. Suppose now that S' moves with a velocity $\mathbf{v}$ in an arbitrary direction relative to S (again assuming that the origins coincide at $t = t' = 0$ and that the axes in the two systems are parallel). We can calculate the corresponding general Lorentz transformation by introducing two additional frames S_0 and S_0' rotated with respect to S and S', respectively, such that $\mathbf{v}$ lies along the z_0 and z_0' axes. We first apply the special Lorentz transformation between S_0 and S_0', and then undo the rotations to recover the transformation between S and S'. The calculation is straightforward but tedious, and we can infer the form of the general Lorentz transformation more easily by arguing along a somewhat different line.

Adopting for the moment the three-vector notation $\mathbf{x} = (x, y, z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\boldsymbol{\beta} = \mathbf{v}/c$, we notice that for the particular choice $\mathbf{v} = (0, 0, v)$ we can write

$$\beta^{-2}\boldsymbol{\beta\beta} \cdot \mathbf{x} = (0, 0, z), \quad (35.29)$$

hence

$$\mathbf{x} - \beta^{-2}\boldsymbol{\beta\beta} \cdot \mathbf{x} = (x, y, 0), \quad (35.30)$$

and therefore

$$[\mathsf{I} + (\gamma - 1)\beta^{-2}\boldsymbol{\beta\beta}] \cdot \mathbf{x} = (x, y, \gamma z). \quad (35.31)$$

Thus the lower right-hand (3×3) matrix in the special Lorentz transformation (35.12) can be regarded as the limiting form of the dyadic $[\mathsf{I} + (\gamma - 1)\beta^{-2}\boldsymbol{\beta\beta}]$. Similarly the row and column three-vectors flanking the

(0, 0) element in (35.12) are limiting forms of the vector $-\boldsymbol{\beta}\gamma$. We therefore propose that the general Lorentz transformation can be written in the form

$$\begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta} \\ -\gamma\boldsymbol{\beta} & \mathsf{I}+(\gamma-1)\beta^{-2}\boldsymbol{\beta\beta} \end{pmatrix}\begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} = \mathbf{L}\begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}, \tag{35.32}$$

or, equivalently,

$$t' = \gamma(t - \mathbf{v}\cdot\mathbf{x}/c^2) \tag{35.33}$$

and

$$\mathbf{x}' = \mathbf{x} + \{[(\gamma-1)(\mathbf{v}\cdot\mathbf{x})/v^2] - \gamma t\}\mathbf{v}. \tag{35.34}$$

We shall find these results useful later.

Equation (35.32) can be verified by the direct calculation mentioned above. It is also easy to show by direct calculation that this transformation satisfies the basic requirement (35.27). Finally, by a lengthy and tedious calculation, one can show that $|\mathbf{L}| = 1$, as required. Thus (35.32) does indeed give the correct Lorentz transformation. The matrix $\mathbf{L}$ is often called the *boost matrix*.

FOUR-VECTORS

We have thus far considered only transformations of the coordinates or of coordinate increments. But clearly we can adopt the Lorentz transformation as a general transformation for any four-vector in spacetime. Thus let A^α be a general contravariant four-vector; often it is convenient to represent it as

$$A^\alpha = (A^{(0)}, \mathbf{a}), \tag{35.35}$$

where $\mathbf{a}$ is an ordinary three-vector composed of the three space components $(A^{(1)}, A^{(2)}, A^{(3)})$ of $\mathbf{A}$. Using the standard transformation law for contravariant vectors, we find that under Lorentz transformation $\mathbf{A}$ becomes $\mathbf{A}'$ where

$$A'^\alpha = (\partial x'^\alpha/\partial x^\beta)A^\beta = L^{\alpha'}_{\beta} A^\beta. \tag{35.36}$$

In matrix notation, which is convenient for calculation, $\mathbf{A}' = \mathbf{LA}$ and $\mathbf{A} = \mathbf{L}^{-1}\mathbf{A}'$ where $\mathbf{A}$ and $\mathbf{A}'$ now denote four-element column vectors.

We can define the general covariant vector B_α as

$$B_\alpha = \eta_{\alpha\beta}B^\beta. \tag{35.37}$$

We then have $B_0 = -B^{(0)}$, $B_1 = B^{(1)}$, $B_2 = B^{(2)}$, and $B_3 = B^{(3)}$ or

$$B_\alpha = (-B^{(0)}, \mathbf{b}). \tag{35.38}$$

Using the standard transformation law for covariant vectors, we find that under Lorentz transformation, $\mathbf{B}$ becomes $\mathbf{B}'$ where

$$B'_\alpha = (\partial x^\beta/\partial x'^\alpha)B_\beta = L^{\beta}_{\alpha'} B_\beta. \tag{35.39}$$

In matrix notation $\mathbf{B}''=\mathbf{B}'\mathbf{L}^{-1}$, or transposing and recalling that $\mathbf{L}^{-1}$ is symmetric, $\mathbf{B}'=\mathbf{L}^{-1}\mathbf{B}$.

We can assign a length l to a four-vector through the relation

$$l^2=\eta_{\alpha\beta}A^\alpha A^\beta=\eta^{\alpha\beta}A_\alpha A_\beta=A^\alpha A_\alpha; \tag{35.40}$$

the result is manifestly an invariant, hence we can uniquely classify general four-vectors as spacelike, timelike, or null.

Second-order contravariant four-tensors transform as

$$T'^{\alpha\beta}=(\partial x'^\alpha/\partial x^\mu)(\partial x'^\beta/\partial x^\nu)T^{\mu\nu}=L^{\alpha'}_{\mu}L^{\beta'}_{\nu}T^{\mu\nu}, \tag{35.41}$$

which in matrix notation is $\mathsf{T}'=\mathbf{L}\mathsf{T}\mathbf{L}'=\mathbf{L}\mathsf{T}\mathbf{L}$ because $\mathbf{L}$ is symmetric. Similarly

$$T^{\alpha\beta}=(\partial x^\alpha/\partial x'^\mu)(\partial x^\beta/\partial x'^\nu)T'^{\mu\nu}=L^{\alpha}_{\mu'}L^{\beta}_{\nu'}T'^{\mu\nu}, \tag{35.42}$$

or $\mathsf{T}=\mathbf{L}^{-1}\mathsf{T}'(\mathbf{L}^{-1})'=\mathbf{L}^{-1}\mathsf{T}\mathbf{L}^{-1}$ because $\mathbf{L}^{-1}$ is symmetric. Similar relations can be written for second-order covariant tensors.

From the above considerations we see that Lorentz transformations of four-vectors and four-tensors are merely special examples of the standard tensor formalism outlined in the appendix. It follows that physical laws expressed as tensor equations containing four-vectors and four-tensors will automatically be covariant under Lorentz transformation, and hence will satisfy Einstein's principle of relativity. We shall therefore seek relativistic generalizations of familiar nonrelativistic relations by attempting to restate them in four-tensor form.

MINKOWSKI COORDINATES

An interesting perspective on the geometrical nature of Lorentz transformation can be obtained by considering the properties of spacetime in a coordinate system introduced by H. Minkowski. Minkowski's formalism was of great importance in the development of relativity theory, and often provides an effective tool for deepening one's understanding of physical problems, particularly in relativistic kinematics.

Minkowski chose the coordinate system $(x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)})=(ict, x, y, z)$, for which $ds^2=\mu_{\alpha\beta}\,dx^\alpha\,dx^\beta$; the *Minkowski metric* is $\mu_{\alpha\beta}=\delta_{\alpha\beta}$. Thus, in this coordinate system, the metric is positive definite, providing an obvious similarity between the spacetime interval and the line element of three-space. In Minkowski coordinates we have $\mathbf{x}'=\mathbf{\Lambda x}$, where the Lorentz transformation is now given by

$$\mathbf{\Lambda}=\begin{pmatrix}\gamma & 0 & 0 & -i\beta\gamma\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ i\beta\gamma & 0 & 0 & \gamma\end{pmatrix}. \tag{35.43}$$

We see that Λ is *Hermitian*, that is, $\Lambda=\Lambda^\dagger$ where "†" denotes the *adjoint* (i.e., conjugate transpose) *matrix* [see, e.g., (**M5,** §§49–51)]. More important, it is easily shown that $\mathbf{\Lambda}^{-1} = \mathbf{\Lambda}^t$, so that in Minkowski coordinates the Lorentz transformation is *orthogonal* (more precisely it is *unitary*). Moreover one easily sees that $|\mathbf{\Lambda}| = 1$. Hence we conclude that in Minkowski coordinates a Lorentz transformation is merely a *rotation* in spacetime. This result applies not only to four-vectors but also to four-tensors; for example it is easy to show that the transformation law for a second-order contravariant tensor becomes $\mathbf{T}' = \mathbf{\Lambda}\mathbf{T}\mathbf{\Lambda}^{-1}$, that is, it is a *similarity transformation*, as expected for a rotation.

Given Minkowski's interpretation of the geometrical significance of the Lorentz transformation as a pure rotation, it becomes intuitively obvious that the form (and value) of the spacetime interval must remain invariant. Even though Minkowski's approach can sometimes provide clear and beautiful insights, in practice it is a nuisance to work with complex vectors and transformation matrices; therefore, having considered the conceptually important results just discussed, we will not use these coordinates further but will work exclusively with the Lorentz metric.

36. *Relativistic Kinematics of Point Particles*

FOUR-VELOCITY

To treat the relativistic kinematics of particles we must develop four-dimensional generalizations of the concepts of velocity and acceleration. If a particle moves on some path $[x(t), y(t), z(t)]$ in three-space, then its three-velocity has components $\mathbf{v} = (dx/dt, dy/dt, dz/dt)$. We cannot use this definition in four-space because t is not invariant under Lorentz transformation; hence the vector $\mathbf{v}$ will not have the transformation properties appropriate to a four-vector. We overcome this difficulty by using the proper time τ as the independent variable because $d\tau$ is a world scalar.

From (35.23) we see that the relationship between proper time and lab-frame time for a particle moving with an instantaneous three-velocity $\mathbf{v}$ is

$$d\tau = \left\{1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\right\}^{1/2} dt = (1 - v^2/c^2)^{1/2}\, dt. \quad (36.1)$$

Clearly proper time is the time measured by a clock in the frame in which the particle is always at rest, the *proper frame* (or *comoving* frame). We will usually distinguish *proper quantities* (i.e., those measured in the proper frame) with a subscript zero, but in the case of proper time it is convenient to use a special symbol. It should be noted that a particle's comoving frame is not generally an inertial frame, a point to which we will return in our discussion of fluid flow.

Given that proper time is a world scalar, we define the *four-velocity* as

$$V^\alpha \equiv (dx^\alpha/d\tau); \quad (36.2)$$

it is obvious that V^α is a genuine contravariant four-vector whose space components reduce to the ordinary three-velocity $\mathbf{v}$ in the limit that $v/c \ll 1$. Hence V^α is indeed the correct four-dimensional generalization of $\mathbf{v}$. In components,

$$V^{(0)} = c(dt/d\tau) = \gamma c, \tag{36.3a}$$

and

$$V^i = (dt/d\tau)(dx^i/dt) = \gamma v^i, \qquad (i = 1, 2, 3), \tag{36.3b}$$

hence

$$V^\alpha = \gamma(c, \mathbf{v}). \tag{36.4}$$

By a trivial calculation one sees that the four-velocity has a constant magnitude:

$$V_\alpha V^\alpha = -c^2. \tag{36.5}$$

Equation (36.5) shows that V^α is timelike. Indeed, the four-velocity of a particle evaluated in its proper frame is $(V^\alpha)_0 = (c, 0, 0, 0)$, hence at any chosen location, $(V^\alpha)_0$ defines the direction of the local proper-time axis in four-dimensional spacetime.

FOUR-ACCELERATION

In ordinary three-space the acceleration is $\mathbf{a} = (d\mathbf{v}/dt)$. Again this expression is not Lorentz covariant, but if we define

$$A^\alpha \equiv (dV^\alpha/d\tau), \tag{36.6}$$

then A^α is a contravariant four-vector whose space components reduce to the ordinary three-acceleration $\mathbf{a}$ in the limit $v/c \ll 1$. Differentiating (36.5) with respect to proper time we find

$$[d(V_\alpha V^\alpha)/d\tau] = 2V_\alpha(dV^\alpha/d\tau) \equiv 0, \tag{36.7}$$

hence

$$V_\alpha A^\alpha \equiv 0, \tag{36.8}$$

which shows that the four-acceleration is orthogonal to the four-velocity in spacetime. Using the easily derived expression

$$(d\gamma/dt) = \gamma^3 \mathbf{v} \cdot \mathbf{a}/c^2, \tag{36.9}$$

we can obtain an explicit expression for A^α in terms of the three-acceleration $\mathbf{a}$, namely

$$A^\alpha = \gamma \frac{d}{dt}[\gamma(c, \mathbf{v})] = \gamma^2 \left[\frac{\gamma^2 \mathbf{v} \cdot \mathbf{a}}{c}, \mathbf{a} + \left(\frac{\gamma^2 \mathbf{v} \cdot \mathbf{a}}{c^2}\right)\mathbf{v}\right], \tag{36.10}$$

and thus the magnitude of A^α is given by

$$A^2 = A_\alpha A^\alpha = \gamma^4[a^2 + (\gamma \mathbf{v} \cdot \mathbf{a}/c)^2], \tag{36.11}$$

which shows that the four-acceleration is spacelike.

37. *Relativistic Dynamics of Point Particles*

FOUR-MOMENTUM

The Newtonian momentum of a particle is $\mathbf{p}=m\mathbf{v}$, where $\mathbf{v}$ is the particle's velocity, and m is its mass, which is assumed to be a unique constant that depends only on the internal constitution of the particle. A relativistic generalization of this expression must replace $\mathbf{v}$ with the four-velocity V^α, and, if the *four-momentum* P^α is to be a genuine four-vector, we can at most multiply V^α by a world scalar. We thus adopt the definition

$$P^\alpha = m_0 V^\alpha, \tag{37.1}$$

where m_0 is the *proper mass* (or *rest mass*) of the particle, that is, its mass in a frame in which it is at rest.

In view of (36.3), equation (37.1) can be written as

$$P^{(0)} = \gamma m_0 c \tag{37.2a}$$

and

$$P^i = \gamma m_0 v^i, \qquad (i=1,2,3), \tag{37.2b}$$

or, more compactly,

$$P^\alpha = \gamma m_0(c, \mathbf{v}). \tag{37.3}$$

We can recover the usual Newtonian definition of momentum from the space components of P^α if we define the *relativistic mass* (or *relative mass*) to be

$$m \equiv \gamma m_0 = m_0/(1-v^2/c^2)^{1/2} \tag{37.4}$$

and write

$$P^\alpha = m(c,\mathbf{v}) = (mc, \mathbf{p}). \tag{37.5}$$

Although it would take us too far afield to discuss this point in detail, one should realize that the concept of rest mass is nontrivial. It implies that we can resolve a real particle into basic constituents (e.g., electrons, protons, . . . , quarks, . . .) that we can *count*, and to each of which we can assign an intrinsic property called "mass" on the basis of *ab initio* calculation using a fundamental theory (e.g., quantum electrodynamics, quantum chromodynamics, . . .) and/or by astute experimentation. This view is quite different from the macroscopic approach in which mass is typically defined operationally by means of experiments (e.g., collisions) that actually deal with particle *momenta* [see e.g., (**L2,** §16)]. Thus while it is often stated that (37.4) implies that the mass of a particle varies with its velocity, we consider this view to be misleading at best (and perhaps flatly wrong), arising from a prerelativistic lack of appreciation for the intrinsically four-dimensional nature of spacetime. We adopt the position that the "real" mass of a particle is its rest mass, and that the variation predicted by (37.4) reflects not a property of matter itself, but rather the variation of its dynamical *effect* [as judged by measurements of particle momentum (and energy—see below)] produced by the relativistic relationship between space and time that results when they are unified into spacetime.

In Newtonian mechanics we demand that the sum $\sum_i \mathbf{p}_i$ of the momenta of all particles be conserved in a system of particles not subject to external forces. The relativistic generalization of this conservation law is that the sum of the particles' four-momenta be conserved. We then have four conservation equations instead of three; as we will soon see, the fourth is a statement of conservation of energy, because the time-component $P^{(0)}$ of the four-momentum is proportional to a particle's energy.

FOUR-FORCE

From Newton's second law we have the prerelativistic relation

$$\boldsymbol{\phi} = (d\mathbf{p}/dt) = d(m\mathbf{v})/dt, \tag{37.6}$$

where $\boldsymbol{\phi}$ is the ordinary three-force acting on a particle. The natural covariant generalization of (37.6) is

$$\Phi^\alpha = (dP^\alpha/d\tau) = m_0(dV^\alpha/d\tau) = m_0 A^\alpha. \tag{37.7}$$

Equivalently,

$$\Phi^\alpha = \left(\frac{dt}{d\tau}\right)\left(\frac{dP^\alpha}{dt}\right) = \gamma \frac{d}{dt}(mc, \mathbf{p}) = \gamma(\dot{m}c, \boldsymbol{\phi}). \tag{37.8}$$

From (37.7) and (36.8) we see that

$$V_\alpha \Phi^\alpha \equiv 0, \tag{37.9}$$

which implies that

$$c^2 \dot{m} = \boldsymbol{\phi} \cdot \mathbf{v} \tag{37.10}$$

and hence

$$\Phi^\alpha = \gamma(\boldsymbol{\phi} \cdot \mathbf{v}/c, \boldsymbol{\phi}). \tag{37.11}$$

Thus the space components of the four-force are proportional to the ordinary (Newtonian) force acting on the particle, while the time component is proportional to $1/c$ times the work done by that force. All four components reduce to their Newtonian values in the limit of small particle velocities. Comparing (37.11) with (36.10) we see that

$$\boldsymbol{\phi} = m[\mathbf{a} + (\gamma^2 \mathbf{v} \cdot \mathbf{a}/c^2)\mathbf{v}], \tag{37.12}$$

which shows that because of relativistic effects the acceleration of a particle is generally not in the same direction as the applied force.

If we evaluate Φ^α in a particle's comoving frame, denoting the value in that frame as Φ_0^α, then we see that $\Phi_0^0 \equiv 0$. This result, which we use again in Chapter 7, is true for all ordinary body forces (e.g., gravity, ignoring general relativistic effects) that act on point particles without changing their internal state. We shall see below how this result must be modified if the internal structure (e.g., chemical or nuclear composition, or internal excitation state) of the particle is affected by the forces acting on it.

ENERGY

Classically, the rate at which the force $\boldsymbol{\phi}$ does work on a particle equals the rate of increase of its kinetic energy $T=\frac{1}{2}mv^2$. But from (37.10) we see that if the rate of work done is to be identified with the rate of increase of a particle's energy, its total energy must be defined to be $\tilde{e}=mc^2+\text{constant}$. The zero point cannot be determined by experiment, so we adopt as the relativistic expression of a particle's energy the relation

$$\tilde{e}=mc^2=\gamma m_0c^2=m_0c^2/(1-v^2/c^2)^{1/2}, \tag{37.13}$$

For $v/c \ll 1$ we can expand (37.13) as

$$\tilde{e}=m_0c^2+\tfrac{1}{2}m_0v^2[1+\tfrac{3}{4}(v^2/c^2)+\tfrac{5}{8}(v^4/c^4)+\ldots]. \tag{37.14}$$

The first component of the second term on the right-hand side of (37.14) is the classical Newtonian formula for kinetic energy, and subsequent terms are relativistic corrections to this formula.

The first term on the right-hand side of (37.14) is nonclassical and implies that matter has a *rest energy* m_0c^2 associated with its rest mass. The rest energy of matter is enormous (9×10^{20} ergs/gram), dwarfing ordinary chemical energies (i.e., excitation, ionization) by many orders of magnitude; for example, the ionization energy of hydrogen corresponds to about 10^{13} ergs/gram. It is this equivalence of mass and energy that results in the release of vast amounts of electromagnetic and thermal energy in stellar interiors, in nuclear explosions, and in power reactors of various kinds, via nuclear reactions that yield products having slightly lower rest masses than the original input nuclei. More to the point, Einstein emphasized that *all* forms of energy (mechanical, thermal, electromagnetic, nuclear, etc.) affect the mass, and hence inertia, of a particle, and that energy itself has inertia.

Using (37.13) we can rewrite the four-momentum given by (37.5) as

$$P^\alpha=(\tilde{e}/c,\mathbf{p}), \tag{37.15}$$

which shows explicitly that conservation of four-momentum in a system of particles implies both energy and momentum conservation, as stated earlier. From (36.5) and (37.1) it follows that

$$P_\alpha P^\alpha=-m_0^2c^2, \tag{37.16}$$

hence from (37.15) we obtain the important formula

$$\tilde{e}^2=p^2c^2+m_0^2c^4. \tag{37.17}$$

Similarly, using (37.13) in (37.8), we can write the four-force as

$$\Phi^\alpha=\gamma(\dot{\tilde{e}}/c,\boldsymbol{\phi}), \tag{37.18}$$

a form that will prove useful below.

We have thus far tacitly assumed that the rest mass of a particle is a strict invariant, and insofar as the particle is truly elementary we can defend this assumption along the lines advanced in our discussion of four-momentum. Now suppose that the "particle" is not elementary, but has an internal structure into which energy can be fed. For example, suppose the particle is an atom, which can absorb or emit energy; in this case one may take the view that the proper mass of our nonelementary particle changes by the mass-equivalent of the energy absorbed or emitted. We must then modify equations (37.7) to (37.11).

For a variable rest mass the four-force becomes

$$\Phi^\alpha = m_0(dV^\alpha/d\tau) + (dm_0/d\tau)V^\alpha. \tag{37.19}$$

Equation (36.8) remains valid, hence we find

$$V_\alpha \Phi^\alpha = -c^2(dm_0/d\tau), \tag{37.20}$$

which shows that V^α and Φ^α are no longer orthogonal. Substituting from (36.4) and (37.18) we find the modified rate-of-work equation

$$\dot{\tilde{e}} = \boldsymbol{\phi} \cdot \mathbf{v} + (c^2 - v^2)(dm_0/d\tau). \tag{37.21}$$

We interpret this equation physically as (rate of increase of particle energy) = (rate of work done by the applied force) + (rate of energy exchange through other, say radiative, mechanisms). We thus deduce that the rate of energy input from nonmechanical sources is

$$W = (c^2 - v^2)(dm_0/d\tau), \tag{37.22}$$

which, evaluated in the comoving frame of the particle, is $W_0 = c^2(dm_0/d\tau)$. We must therefore modify (37.11) to read

$$\Phi^\alpha = \gamma[(\boldsymbol{\phi} \cdot \mathbf{v} + W)/c, \boldsymbol{\phi}], \tag{37.23}$$

which shows that in the presence of nonmechanical effects that modify the internal state of the "particle", we can no longer assume that $\Phi_0^0 \equiv 0$; rather, these effects perform "work" as if some additional "force" were acting. We will find terms of precisely this kind in the equations of radiation hydrodynamics discussed in §§93 and 96.

PHOTONS

Suppose we choose photons as the particles to be considered. From quantum mechanics we know that a photon's energy is $\tilde{e} = h\nu$ and its momentum is $p = h\nu/c$, where ν is its frequency. From (37.17) we see that $(m_0)_{\text{photon}} \equiv 0$, that is, *photons have zero rest mass.* Because a photon's rest mass vanishes the definition of four-momentum given by (37.1) is no longer useful.

Nonetheless equation (37.15) remains valid, hence we can write the photon four-momentum as

$$M^\alpha = (P^\alpha)_{\text{photon}} = (h\nu/c)(1, \mathbf{n}) = \hbar(k, \mathbf{k}) \equiv \hbar K^\alpha, \tag{37.24}$$

where $\mathbf{n}$ is the unit vector in the photon's direction of propagation, and

$k = (2\pi\nu/c) = (2\pi/\Lambda)$ is the photon's *wavenumber*. In (37.24), K^α is the *photon-propagation four-vector*, which, as one expects, is a null vector,

$$K_\alpha K^\alpha \equiv 0, \tag{37.25}$$

as is the photon four-momentum

$$M_\alpha M^\alpha \equiv 0. \tag{37.26}$$

4.2 Relativistic Dynamics of Ideal Fluids

Let us now consider the relativistic dynamics of a compressible ideal fluid, that is, we ignore, for the present, viscous and conduction effects. In §§38–42 we view the gas as a continuum; even in this case it will sometimes be convenient to use particle-counting arguments. We adopt a kinetic-theory view in §43.

38. Kinematics

In seeking relativistic generalizations of the usual classical expressions that describe the kinematics of a fluid, let us first consider the question of reference frames. In general, the velocity of a fluid as measured in a fixed laboratory frame is a function of both space and time: $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$. Therefore when we speak of the *comoving* (or *proper*) *frame* of a fluid parcel, we are in general dealing with a non-inertial frame, because the fluid can accelerate as it moves. In what follows, we need to apply Lorentz transformations to relate quantities measured in the comoving frame to those measured in the laboratory frame. Here we encounter a problem because, strictly speaking, the Lorentz transformation applies only between inertial frames, which have a constant velocity with respect to one another. To deal with this problem, special relativity hypothesizes that we can consider the comoving frame for any particular fluid parcel to comprise a *sequence* of inertial frames, each of which has a velocity instantaneously coinciding with that of the fluid parcel; it is then assumed that a Lorentz transformation applies between each of these inertial frames and the lab frame. When this is done, the resulting formulation is internally consistent and yields results in agreement with experiment. Further discussion of this point can be found in, for example, (**S1,** Chapters 4 and 6), and the references cited therein.

How do we now describe the motion of a fluid and its time evolution? The covariant generalizations to be used for the velocity and acceleration of a fluid element are, of course, its four-velocity and four-acceleration. To describe time evolution, we must develop a covariant form for the Lagrangean time derivative (D/Dt). From a Newtonian view, (D/Dt) is the time derivative evaluated following the motion of the fluid; put another way, it is $(D/Dt)_0$, the time derivative evaluated in the comoving frame, which, in relativistic terms, is just the derivative with respect to proper time

$(D/D\tau)$. We thus generalize the Lagrangean derivative to mean $(D/D\tau)$, which is a scalar operator in spacetime because proper time is a world scalar.

It is easy to write an expression for $(D/D\tau)$, namely

$$\frac{D}{D\tau}=\left(\frac{dt}{d\tau}\right)\frac{\partial}{\partial t}+\left(\frac{dx^i}{d\tau}\right)\frac{\partial}{\partial x^i}=V^0\frac{1}{c}\frac{\partial}{\partial t}+V^i\frac{\partial}{\partial x^i}, \tag{38.1}$$

or, more compactly,

$$(D/D\tau)=V^\alpha(\partial/\partial x^\alpha), \tag{38.2}$$

which is manifestly Lorentz covariant. In the limit $v/c\ll 1$, $(D/D\tau)$ clearly reduces to the Lagrangean (D/Dt).

As written, (38.2) applies only in Cartesian coordinates in a flat spacetime. We can easily generalize to curvilinear coordinates in flat spacetime ["flat" implying that all components of the Riemann curvature tensor are zero; see, for example, (**A1,** 149) or (**M3,** 283)] by replacing the ordinary derivatives with covariant derivatives to obtain

$$(Df/D\tau)=V^\alpha f_{;\alpha}. \tag{38.3}$$

Here f is any differentiable function. From (38.3) we recognize that $(D/D\tau)$ is the *intrinsic derivative* with respect to proper time in a four-dimensional spacetime (cf. §15 and §A3.10). If spacetime is indeed flat then the covariant derivative in (38.3) merely accounts for curvature of the three-space coordinate mesh (say spherical coordinates) in the "ordinary space" part of the Lorentz metric. But it is worth mentioning that (38.3) is also valid in curved spacetime for general line elements of the form $ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu$, where $g_{\mu\nu}=g_{\mu\nu}(x^{(0)},x^{(1)},x^{(2)},x^{(3)})$; hence (38.3) also applies in general relativity.

39. The Equation of Continuity

In Newtonian hydrodynamics the density is the mass per unit volume, or the number of particles per unit volume times the mass of each particle; because both of these quantities are presumed to be invariants, the Newtonian density is considered to be the same in all frames (e.g., in the laboratory and comoving frames). When relativistic effects become important, however, the situation is more complicated, and several definitions of density, each useful in certain contexts, can be made.

First, suppose that in the comoving frame we have N_0 particles per unit proper volume, each of proper mass m_0; then the *density of proper mass in the comoving (proper) frame* is

$$\rho_0=N_0m_0. \tag{39.1}$$

As measured in the laboratory frame, the density of proper mass will be different. If we choose a comoving volume element δV_0, then the *number* of particles in it will be $N_0\,\delta V_0$; if we count the same particles in the lab frame we will, of course, get the same number, hence

$$N\,\delta V = N_0\,\delta V_0, \tag{39.2}$$

where N denotes the lab-frame particle density and δV is the lab-frame volume element corresponding to δV_0. But owing to Lorentz contraction [cf. (35.20)],

$$\delta V = \delta V_0/\gamma \tag{39.3}$$

hence

$$N = \gamma N_0. \tag{39.4}$$

Therefore the *density of proper mass in the laboratory frame* is

$$\rho = N m_0 = \gamma \rho_0. \tag{39.5}$$

From the Newtonian view this quantity can be considered, as our choice of notation implies, to be "the" density. It is also sometimes useful to define the *density of relative mass* as measured in the lab frame to be

$$\rho' = Nm = \gamma^2 N_0 m_0 = \gamma\rho = \gamma^2 \rho_0, \tag{39.6}$$

in terms of which we can write the momentum density as $\rho'\mathbf{v}$, where $\mathbf{v}$ is the ordinary three-velocity of the fluid.

To derive a relativistic version of the equation of continuity we start from the standard Newtonian equation

$$\rho_{,t} + (\rho v^i)_{,i} = 0, \tag{39.7}$$

which, using (39.5), can be rewritten as

$$(\gamma\rho_0)_{,t} + (\gamma\rho_0 v^i)_{,i} = 0. \tag{39.8}$$

Now recalling that $V^\alpha = \gamma(c, \mathbf{v})$, we see that (39.8) is simply

$$(\rho_0 V^\alpha)_{,\alpha} = 0, \tag{39.9}$$

which is manifestly covariant under Lorentz transformation. We thus accept (39.9) as the correct covariant generalization of the equation of continuity in a flat spacetime.

If we choose curvilinear coordinates or have a curved spacetime, the further generalization of (39.9) is

$$(\rho_0 V^\alpha)_{;\alpha} = 0. \tag{39.10}$$

For example, see (**P1**, 230) for (39.10) written in spherical coordinates.

The vector $N_0 V^\alpha$ is the four-dimensional *particle flux density vector*. The equation of continuity is thus merely a statement that the particle flux density is conserved in spacetime, that is, that particles are neither created nor destroyed. If particles are *not* conserved (e.g., nuclear reactions occur), then a source-sink term must appear on the right-hand sides of (39.9) and (39.10).

40. The Material Stress-Energy Tensor

In §23 we saw that in Newtonian fluid dynamics the three equations of momentum conservation can be formulated as

$$(\rho v^i)_{,t} + \Pi^{ij}_{,j} = f^i, \tag{40.1}$$

where $\mathbf{\Pi}$ is the momentum flux density tensor

$$\Pi^{ij} \equiv \rho v^i v^j + p\,\delta^{ij}, \tag{40.2}$$

and $\mathbf{f}^i$ is the externally applied force density. We wish to construct a similar formulation in spacetime, and we therefore seek four-dimensional generalizations of $\mathbf{\Pi}$ and $\mathbf{f}$. We defer the question of the force density to §41 and concentrate here on obtaining an appropriate expression for the *material stress-energy tensor* M, which is the Lorentz-covariant generalization of the Newtonian momentum flux density tensor. Notice also that the left-hand side of each equation in (40.1) can be expressed as the four-divergence of a suitable four-vector, hence we expect to be able to cast the dynamical equations into the form of a four-divergence of M.

It is clear from the outset that we will arrive at four conservation relations rather than three. Of these, three will be momentum conservation equations. From (40.1) we see that the space components M^{ij} of the stress-energy tensor should be generalizations of Π^{ij}, and hence account for the momentum flux density, both macroscopic and microscopic (i.e., pressure), in the fluid, while the zeroth column M^{i0} must reduce to c times a component of the momentum density. [The factor of c is needed because in the four-divergence operator, $(\partial/\partial x^0)$ is $c^{-1}(\partial/\partial t)$.] Furthermore, we recall from §37 that when we generalized the conservation law for the total Newtonian momentum of a group of particles to conservation of their total four-momentum, the fourth equation turned out to be an expression of energy conservation. The same is true for a fluid, hence we expect the stress-energy tensor to contain an element M^{00} representing the total energy density of the fluid and a vector M^{0j} representing $(1/c)$ times the energy flux in the jth direction of the flow. (Again the factor of $1/c$ is needed to balance the same factor in $\partial/\partial x^0$.)

Thus on the basis of the Newtonian equations alone we expect M to be of the general form

$$\mathsf{m} = \begin{pmatrix} \rho_{00}c^2 & c\rho\mathbf{v} \\ c\rho\mathbf{v} & \rho v^i v^j + p\,\delta^{ij} \end{pmatrix} \tag{40.3}$$

where we have written m instead of M to emphasize that (40.3) is not yet the relativistic stress-energy tensor, inasmuch as it is not covariant. Following L. H. Thomas (**T2**), we have written

$$\rho_{00} \equiv \rho_0(1 + e/c^2) \tag{40.4}$$

where e is the specific internal energy of the gas produced by the microscopic motions of its constituent particles, ρ_{00} is the total mass density of the fluid, including the mass equivalent of its thermal energy, and $\rho_{00}c^2$ is the total energy density of the fluid. Both ρ_0 and e are defined in the proper frame (see §43) and are world scalars, as is ρ_{00}.

In constructing an expression for M we are guided by three general principles: (1) it must be covariant, and hence must contain only world scalars, four-vectors, and four-tensors; (2) it must give the correct fluid energy density and hydrostatic pressure in the comoving frame; (3) it must yield the correct nonrelativistic equations in the laboratory frame when $v/c \ll 1$.

From (40.3) with $\mathbf{v} = 0$ we see that in the comoving frame we must have

$$\mathsf{M}_0 = \begin{pmatrix} \rho_{00}c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{40.5}$$

We could now obtain the components of M in the lab frame by applying a Lorentz transformation to (40.5) as in equation (35.42) [see e.g., (**W2,** 48)]. But a simpler approach is to notice that in the comoving frame $V_0^\alpha = (c, 0, 0, 0)$, hence in this frame

$$(V^\alpha V^\beta)_0 = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{40.6}$$

while in this same frame the *projection tensor* is

$$(P^{\alpha\beta})_0 \equiv \eta^{\alpha\beta} + c^{-2}(V^\alpha V^\beta)_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{40.7}$$

We use the projection tensor extensively in §4.3. Substituting (40.6) and (40.7) into (40.5) we conclude that

$$(M^{\alpha\beta})_0 = (\rho_{00} + p/c^2)(V^\alpha V^\beta)_0 + p\eta^{\alpha\beta}. \tag{40.8}$$

Again we follow Thomas (**T2**) and define

$$\rho_{000} \equiv \rho_{00} + p/c^2 = \rho_0(1 + h/c^2), \tag{40.9}$$

where h is the specific enthalpy of the fluid. If we can assume that p is a world scalar (see §43) then we see that

$$M^{\alpha\beta} = \rho_{000} V^\alpha V^\beta + p\eta^{\alpha\beta} \tag{40.10}$$

is a fully covariant expression for M, which reduces to (40.8) in the comoving frame. Notice that M is symmetric.

It is obvious by inspection that (40.10) satisfies requirements (1) and (2) stated above. To check whether it also satisfies requirement (3), we write $\rho = \gamma\rho_0$ as in (39.5), note that in a nonrelativistic fluid $e/c^2 \ll 1$ and $p/\rho_0 c^2 \ll 1$, and expand in powers of v/c. We find

$$\begin{aligned} M^{ij} &= \gamma^2 \rho_0 v^i v^j [1 + (\rho_0 e + p)/\rho_0 c^2] + p\,\delta^{ij} \rightarrow \rho v^i v^j + p\,\delta^{ij} + O(v^2/c^2) \\ &= \Pi^{ij} + O(v^2/c^2), \end{aligned} \tag{40.11}$$

which is the correct nonrelativistic expression, while

$$M^{0i} = M^{i0} = \gamma^2 \rho_0 c v^i [1 + (\rho_0 e + p)/\rho_0 c^2] \rightarrow c\rho v^i + O(v^2/c^2), \tag{40.12}$$

which is c times the momentum density, as expected. Furthermore,

$$\begin{aligned} M^{00} &= \gamma^2 \rho_0 (c^2 + e + p v^2/\rho_0 c^2) \\ &= \gamma\rho(c^2 + e) + O(v^2/c^2) \rightarrow \rho c^2 + \tfrac{1}{2}\rho v^2 + \rho e + O(v^2/c^2), \end{aligned} \tag{40.13}$$

which is the correct nonrelativistic energy density (including rest energy) of the fluid. Hence (40.10) does in fact provide a fully satisfactory expression for M.

The stress-energy tensor does not in itself provide a complete description of the fluid. To obtain a complete system of equations we also require constitutive relations that describe the microphysics of the gas. We need at least a caloric equation of state relating e to p and ρ_0, and perhaps also an equation of state of the form $p = p(\rho_0, T)$ where T is the *proper temperature* of the fluid. In practice these relations must be obtained from microscopic kinetic-theoretic considerations, and are operationally definable only in the comoving frame of the fluid (see §43).

Finally, it is worth noting that (40.10) also applies in general relativity if we replace the Lorentz metric $\eta^{\alpha\beta}$ with a general metric $g^{\alpha\beta}$.

41. The Four-Force Density

To obtain a Lorentz-covariant generalization of the right-hand side of (40.1) we use the *four-force density*

$$F^\mu \equiv \Phi^\mu/\delta V_0 = (\gamma/\delta V_0)(\boldsymbol{\phi}\cdot\mathbf{v}/c, \boldsymbol{\phi}), \tag{41.1}$$

where Φ^μ and $\boldsymbol{\phi}$ are, respectively, the four-force and the Newtonian force acting on a finite element of material contained in the proper volume δV_0. F^μ is a four-vector because Φ^μ is a four-vector and δV_0 is a world scalar. In any arbitrary frame we define the *ordinary force density* to be $\mathbf{f} = \boldsymbol{\phi}/\delta V$. Therefore

$$F^\mu = (\gamma\,\delta V/\delta V_0)(\mathbf{f}\cdot\mathbf{v}/c, \mathbf{f}). \tag{41.2}$$

But from (39.3) $\delta V_0 = \gamma\,\delta V$, hence

$$F^\mu = (\mathbf{f}\cdot\mathbf{v}/c, \mathbf{f}). \tag{41.3}$$

Thus in the relativistic equations of hydrodynamics the four-force density in any frame has space components equal to the Newtonian force density in that frame, and has a time component equal to the rate of work, per unit volume, done by the Newtonian force density. Put another way, the four-force density has space components equal to the rate of increase (per unit volume) of the momentum, and a time component equal to $1/c$ times the rate of increase of the energy, of the material, as measured in the frame of reference adopted.

42. The Dynamical Equations

GENERAL FORM

Arguing by analogy to the Newtonian equations (40.1), we expect the relativistic fluid-dynamical equations to have the general form

$$M^{\alpha\beta}_{;\beta} = F^{\alpha}, \tag{42.1}$$

or

$$(\rho_{000} V^{\alpha} V^{\beta} + p g^{\alpha\beta})_{;\beta} = F^{\alpha}. \tag{42.2}$$

Here $g^{\alpha\beta}$ may be the Lorentz metric $\eta^{\alpha\beta}$ if we use Cartesian coordinates in a flat spacetime, may have space components appropriate to curvilinear coordinates imbedded in a flat spacetime, or may be a general metric in curved spacetime.

Writing (42.1) and (42.2) out in Cartesian coordinates, using (40.10) and (41.3), and defining $\rho_1 \equiv \gamma^2 \rho_{000}$, we obtain

$$(\rho_1 - p/c^2)_{,t} + (\rho_1 v^j)_{,j} = f^0/c = (\mathbf{f} \cdot \mathbf{v}/c^2), \tag{42.3}$$

and

$$(\rho_1 v^i)_{,t} + (\rho_1 v^i v^j + p\,\delta^{ij})_{,j} = f^i, \qquad (i = 1, 2, 3), \tag{42.4a}$$

or

$$(\rho_1 v_i)_{,t} + (\rho_1 v_i v^j)_{,j} + p_{,i} = f_i, \qquad (i = 1, 2, 3). \tag{42.4b}$$

In this form the equations bear a close resemblance to their Newtonian counterparts. Equations (42.4) are the momentum equations. Equation (42.3), while bearing a superficial resemblance to the Newtonian continuity equation (to which it reduces in the limit $c \to \infty$), is actually the energy equation; this becomes more apparent if we write out ρ_1 to display all physical variables explicitly:

$$[\gamma^2(\rho_0 c^2 + \rho_0 e + p) - p]_{,t} + [\gamma^2(\rho_0 c^2 + \rho_0 e + p) v^j]_{,j} = \mathbf{f} \cdot \mathbf{v}, \tag{42.5}$$

and

$$[\gamma^2(\rho_0 c^2 + \rho_0 e + p) v_i]_{,t} + [\gamma^2(\rho_0 c^2 + \rho_0 e + p) v_i v^j]_{,j} + c^2 p_{,i} = c^2 f_i \qquad (i = 1, 2, 3). \tag{42.6}$$

If in (42.5) one again defines $\rho = \gamma\rho_0$, expands the other factor γ, drops terms of $O(v^2/c^2)$ and higher, and, finally, subtracts c^2 times the continuity equation (39.9), one recovers the nonrelativistic energy equation (24.6). We will discuss a similar reduction of the momentum equation (42.6) shortly.

Detailed expressions for (42.5) and (42.6) in spherical coordinates are given in (**P1,** 230–231).

THE GAS ENERGY EQUATION

We can cast the energy equation (42.3) into a much more revealing form by reducing it to a gas energy equation following L. H. Thomas (**T2**). Form the inner product of (42.2) with V_α to obtain

$$V_\alpha V^\alpha(\rho_{000}V^\beta)_{;\beta}+\rho_{000}V^\beta(V_\alpha V^\alpha_{;\beta})+g^{\alpha\beta}V_\alpha p_{,\beta}=V_\alpha F^\alpha. \qquad (42.7)$$

But $V_\alpha V^\alpha=-c^2$, which implies that $V_\alpha V^\alpha_{;\beta}\equiv 0$; hence (42.7) reduces to

$$c^2(\rho_{000}V^\alpha)_{;\alpha}-V^\alpha p_{,\alpha}=-V_\alpha F^\alpha. \qquad (42.8)$$

Now subtracting (c^2+e+p/ρ_0) times the continuity equation (39.10) from (42.8) we find

$$\rho_0 V^\alpha[(\partial e/\partial x^\alpha)-(p/\rho_0^2)(\partial\rho_0/\partial x^\alpha)]=-V_\alpha F^\alpha, \qquad (42.9)$$

or, recalling that $V^\alpha(\partial/\partial x^\alpha)\equiv(D/D\tau)$,

$$\rho_0\left[\frac{De}{d\tau}+p\frac{D}{D\tau}\left(\frac{1}{\rho_0}\right)\right]=-V_\alpha F^\alpha. \qquad (42.10)$$

From (37.23) and (41.1) it follows that if we are dealing with ordinary body forces, which conserve particle numbers and rest masses, $V_\alpha F^\alpha=0$. The flow is then adiabatic and we obtain

$$(De/D\tau)+p[D(1/\rho_0)/D\tau]=0. \qquad (42.11)$$

as the relativistic generalization of the Newtonian gas-energy equation for an ideal gas. Equation (42.11) is the one that most simply relates e, p, and ρ_0, which are all defined in the comoving frame. Moreover, it obviously reduces to the Newtonian gas-energy equation as $c\to\infty$, and differs from it in general by terms of $O(v^2/c^2)$.

Of course, if physical processes operate in which the numbers or rest masses of the particles in the fluid are not conserved (nuclear reactions), or through which the fluid can dissipate or transport energy internally (viscosity and conductivity), or exchange energy with an external source (radiation), then $V_\alpha F^\alpha\neq 0$, and the flow is no longer adiabatic. We consider such cases in §4.3 and Chapter 7.

THE MOMENTUM EQUATION

The momentum equations (42.4) can also be manipulated into a much more useful form. Multiplying (42.3) by v_i and subtracting the result from (42.4b) we obtain

$$\rho_1(v_{i,t}+v^j v_{i,j})+p_{,i}+(v_i/c^2)p_{,t}=f_i-v_i(\mathbf{f}\cdot\mathbf{v}/c^2). \qquad (42.12)$$

Then using (38.2) for $(D/D\tau)$ and defining $\rho_* \equiv \gamma\rho_{000}$, which is the relative density corresponding to the proper mass density plus the mass equivalent of the fluid enthalpy, we have

$$\rho_*(D\mathbf{v}/D\tau) = \mathbf{f} - \nabla p - c^{-2}\mathbf{v}(p_{,t} + \mathbf{f} \cdot \mathbf{v}), \tag{42.13}$$

which is the relativistic generalization of Euler's equation of motion (23.6) for an ideal fluid. These equations assume their simplest form in the comoving frame where $\mathbf{v} = 0$; we then have

$$\rho_{000}(D\mathbf{v}/D\tau) = \rho_{000}(D\mathbf{v}/Dt)_0 = \mathbf{f}_0 - \nabla p. \tag{42.14}$$

Equation (42.14) now appears almost identical to Euler's equation.

For problems of fluid flow, a characteristic time-increment is $\Delta t \sim \Delta x/v$, where Δx is a characteristic length and v is a typical velocity in the flow. Therefore the term in $(\partial p/\partial t)$ in (42.13) is $O(v^2/c^2)$ compared to ∇p; it is thus apparent that (42.13) differs from its Newtonian counterpart only by terms that are $O(v^2/c^2)$. In Chapter 6 we find that when radiative effects are taken into account the situation is quite different, because frame-dependent terms that are $O(v/c)$ appear.

GENERAL RELATIVISTIC EQUATIONS

In general relativity the hydrodynamic equations can be written in the very compact form

$$M^{\alpha\beta}_{;\beta} = 0. \tag{42.15}$$

Here one assumes that spacetime is Riemannian, with an *intrinsic curvature*, and is described by a general metric $g_{\alpha\beta}$ of which each element can be a function of $x^{(0)}, \ldots, x^{(3)}$. In such a formulation, the quantities interpreted as forces in special relativity are found not to be independent physical entities, but instead are results of spacetime curvature, which, when one computes covariant derivatives in the curved manifold, leads to additional terms interpretable as forces. For a detailed discussion of this view the reader should consult one of the texts on general relativity cited at the end of this chapter.

43. *The Kinetic Theory View*

Instead of considering the fluid to be a continuum, let us now suppose it to be composed of a large number of paricles, each having a rest mass m_0. We can use kinetic theory to calculate particle-energy and particle-momentum densities and fluxes; in doing so, we will find that we have constructed the fluid stress-energy tensor directly from microscopic considerations. Our treatment parallels that given in (**L1,** §10) and (**P1,** Chap. 9) to which the reader can refer for further details. We do not attempt to treat particle collisions or general relativistic effects; these topics are discussed in (**D1**), (**E2**), and (**S2**), and the references cited therein.

THE DISTRIBUTION FUNCTION

Let $f(\mathbf{x}, \mathbf{p}, t)$ be the *particle distribution function* defined such that at time t the number of particles in a volume element $d\mathbf{x}$ centered on $\mathbf{x}$, and with momenta in a momentum-space element $d\mathbf{p}$ centered on $\mathbf{p}$, is $f(\mathbf{x}, \mathbf{p}, t)\, d\mathbf{x}\, d\mathbf{p}$; here all quantities are measured in the laboratory frame. In what follows we will focus on a single point in spacetime; to economize the notation we suppress reference to $\mathbf{x}$ and t.

If a particle's velocity in the lab frame is $\mathbf{u}$, decompose it into

$$\mathbf{u} = \mathbf{v} + \mathbf{U} \tag{43.1}$$

where

$$\mathbf{v} \equiv \int \mathbf{u} f(\mathbf{p})\, d\mathbf{p} \Big/ \int f(\mathbf{p})\, d\mathbf{p} \tag{43.2}$$

is the *flow velocity* of the fluid, that is, the average velocity of the particles in a small neighborhood of $\mathbf{x}$, and $\mathbf{U}$ is a particle's *random velocity* (measured in the lab frame) relative to the flow velocity.

The frame moving with the flow velocity is the comoving frame; quantities measured in this frame are denoted with subscript zero. For example, the random velocity of a particle in this frame is $\mathbf{U}_0$, its momentum is $\mathbf{p}_0$. and the distribution function is $f_0(\mathbf{p}_0)$. The third set of frames of interest comprises the rest frames of the particles, each of which moves with one of the particles. Quantities measured in one of these frames will be denoted with a prime, except for a particle's rest mass, which we will still call m_0.

INVARIANCE OF THE DISTRIBUTION FUNCTION

Suppose we choose a definite group of particles. Observers in both the lab and comoving frames will count the same *number* of particles in the group, even though the particles will be observed to be in different phase-space volume elements. We therefore have

$$f(\mathbf{p})\, d\mathbf{x}\, d\mathbf{p} = f_0(\mathbf{p}_0)\, d\mathbf{x}_0\, d\mathbf{p}_0. \tag{43.3}$$

Consider a proper volume element $d\mathbf{x}'$ in the rest frame of some particle. According to (39.3) an observer in the comoving frame will measure its volume to be

$$d\mathbf{x}_0 = (1 - U_0^2/c^2)^{1/2}\, d\mathbf{x}', \tag{43.4}$$

while an observer in the lab frame will measure its volume as

$$d\mathbf{x} = (1 - u^2/c^2)^{1/2}\, d\mathbf{x}'. \tag{43.5}$$

Hence

$$d\mathbf{x}_0 = \left(\frac{1 - U_0^2/c^2}{1 - u^2/c^2}\right)^{1/2} d\mathbf{x}. \tag{43.6}$$

On the other hand, if the particle has rest mass m_0, (37.13) implies that its energy in the comoving frame is

$$\tilde{e}_0 = m_0 c^2/(1 - U_0^2/c^2)^{1/2}, \tag{43.7}$$

and in the lab frame it is

$$\tilde{e} = m_0 c^2/(1 - u^2/c^2)^{1/2}; \tag{43.8}$$

comparing with (43.6) we conclude that

$$d\mathbf{x}_0 = (\tilde{e}/\tilde{e}_0)\, d\mathbf{x}. \tag{43.9}$$

We can relate $d\mathbf{p}$ to $d\mathbf{p}_0$ by applying the general Lorentz transformation (35.32) with velocity $-\mathbf{v}$ (the velocity of the lab frame as seen from the comoving frame) to a particle's four-momentum measured in the comoving frame. We obtain

$$\tilde{e} = \gamma(\tilde{e}_0 + \mathbf{v} \cdot \mathbf{p}_0), \tag{43.10}$$

and

$$\mathbf{p} = \mathbf{p}_0 + [\gamma(\tilde{e}_0/c^2) + (\gamma - 1)\mathbf{v} \cdot \mathbf{p}_0/v^2]\mathbf{v}. \tag{43.11}$$

In general we can write

$$d\mathbf{p} = J(p^{(1)}, p^{(2)}, p^{(3)}/p_0^{(1)}, p_0^{(2)}, p_0^{(3)})\, d\mathbf{p}_0, \tag{43.12}$$

where J is the Jacobian of the transformation from the comoving system to the lab system. To simplify the calculation we do not use (43.11) directly, but instead rotate the coordinate axes such that in the new coordinate system the comoving frame moves with velocity v along $x^{(3)}$. In this new system we have from (35.15).

$$(\tilde{e}/c, p^{(1)}, p^{(2)}, p^{(3)}) = [\gamma(\tilde{e}_0/c + \beta p_0^{(3)}), p_0^{(1)}, p_0^{(2)}, \gamma(p_0^{(3)} + \beta \tilde{e}_0/c)], \tag{43.13}$$

whence

$$J = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma\left(1 + \dfrac{\beta}{c}\dfrac{\partial \tilde{e}_0}{\partial p_0^{(3)}}\right) \end{vmatrix} = \gamma\left(1 + \frac{\beta}{c}\frac{\partial \tilde{e}_0}{\partial p_0^{(3)}}\right). \tag{43.14}$$

But from (37.17) we know that $\tilde{e}_0^2 = p_0^2 c^2 + m_0^2 c^4$, hence

$$(\partial \tilde{e}_0/\partial p_0^{(3)}) = p_0^{(3)} c^2/\tilde{e}_0, \tag{43.15}$$

and therefore

$$J = \gamma(1 + v p_0^{(3)}/\tilde{e}_0) = \tilde{e}/\tilde{e}_0, \tag{43.16}$$

where the second equality follows from (43.13). The latter expression for J contains no reference to the orientation of the coordinate axes, and hence applies in the original coordinate system as well. Combining (43.12) with (43.16) we have

$$d\mathbf{p} = (\tilde{e}/\tilde{e}_0)\, d\mathbf{p}_0, \tag{43.17}$$

hence from (43.9) we find

$$d\mathbf{x}\, d\mathbf{p} = d\mathbf{x}_0\, d\mathbf{p}_0. \tag{43.18}$$

Using (43.18) in (43.3) we see that the distribution function f is *invariant* under Lorentz transformation, that is,

$$f(\mathbf{p}) = f_0(\mathbf{p}_0). \tag{43.19}$$

THE COMOVING-FRAME NUMBER DENSITY, ENERGY DENSITY, AND PRESSURE

From the point of view of thermodynamics, the fluid energy density and the pressure are best defined in the comoving frame of the fluid. First, we express the comoving-frame *particle number density* as the integral of the distribution function over all momenta:

$$N_0 \equiv \int f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.20}$$

Given N_0, the density of proper mass in the comoving frame is $\rho_0 = N_0 m_0$.

The *total energy density* in the comoving frame is obtained by taking the sum over all momenta of the product of the number of particles at a given momentum times the energy of those particles, that is,

$$\rho_{00} c^2 = \int \tilde{e}_0 f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.21}$$

Here ρ_{00} has the same meaning as in §40. Combined with (40.4) and (43.20), equation (43.21) provides an operational definition of the fluid's specific internal energy e.

The stress tensor in the comoving frame is obtained by calculating the rate of momentum transfer across a unit area, as was done to derive equation (30.15). We find

$$-T^{ij} = \int U_0^i p_0^j f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.22}$$

To recover the correct equations of hydrodynamics for an ideal fluid we must now assume that the distribution function $f_0(\mathbf{p}_0)$ is isotropic in the comoving frame. The stress tensor is then diagonal, $T^{ij} = -p_m\, \delta^{ij}$, where the *pressure* is given by

$$p_m = \int U_0^i p_0^i f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.23}$$

In (43.23) there is no sum on i, and the subscript "m" for "material" has been used to avoid confusion with the magnitude p of the momentum vector $\mathbf{p}$.

Because f_0 is isotropic, we can evaluate (43.23) along any axis. Thus we can equally well write

$$p_m = \int (\mathbf{U}_0 \cdot \mathbf{l})(\mathbf{p}_0 \cdot \mathbf{l}) f_0(\mathbf{p}_0)\, d\mathbf{p}_0 \tag{43.24}$$

where $\mathbf{l}$ is an arbitrarily oriented unit vector; this form will prove useful in what follows.

THE LABORATORY-FRAME PARTICLE DENSITY, ENERGY DENSITY, AND MOMENTUM DENSITY

Let us now calculate the particle density, energy density, and momentum density of the fluid as measured in the laboratory frame. The *particle density* is

$$N = \int f(\mathbf{p})\, d\mathbf{p}. \tag{43.25}$$

Using (43.10), (43.17), and (43.19) we can rewrite (43.25) as

$$N = \gamma \int [1 + (\mathbf{v} \cdot \mathbf{p}_0)/\tilde{e}_0] f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.26}$$

Because f_0 is isotropic the integral containing $(\mathbf{v} \cdot \mathbf{p}_0)$ vanishes, and we have

$$N = \gamma N_0, \tag{43.27}$$

in agreement with (39.4).

The *energy density* (*ED*) in the laboratory frame is

$$ED = \int \tilde{e} f(\mathbf{p})\, d\mathbf{p} = \gamma^2 \int (1/\tilde{e}_0)(\tilde{e}_0 + \mathbf{v} \cdot \mathbf{p}_0)^2 f_0(\mathbf{p}_0)\, d\mathbf{p}_0, \tag{43.28}$$

where we again used (43.10), (43.17), and (43.19). Expanding the square, and again noting that the integral containing $(\mathbf{v} \cdot \mathbf{p}_0)$ vanishes because f_0 is isotropic, we have

$$ED = \gamma^2 \int [\tilde{e}_0 + (\mathbf{v} \cdot \mathbf{p}_0)^2/\tilde{e}_0] f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.29}$$

From (37.3) and (37.13) we have

$$\mathbf{p}_0 = \tilde{e}_0 \mathbf{U}_0/c^2. \tag{43.30}$$

Hence (43.29) can be rewritten as

$$ED = \gamma^2 \int [\tilde{e}_0 + (\mathbf{U}_0 \cdot \mathbf{v})(\mathbf{p}_0 \cdot \mathbf{v})/c^2] f_0(\mathbf{p}_0)\, d\mathbf{p}_0, \tag{43.31}$$

and if we choose $\mathbf{l} = (\mathbf{v}/v)$, we see from (43.21) and (43.24) that

$$ED = \gamma^2[\rho_{00}c^2 + (v^2/c^2)p_m]. \tag{43.32}$$

Using the identity $(v^2/c^2) = 1 - \gamma^{-2}$ we obtain finally

$$ED = \gamma^2(\rho_{00}c^2 + p_m) - p_m = \gamma^2 \rho_{000} c^2 - p_m, \tag{43.33}$$

which is identical to M^{00} in the stress-energy tensor, as expected.

The *momentum density* (*MD*) in the laboratory frame is

$$MD = \int \mathbf{p} f(\mathbf{p})\, d\mathbf{p} = \int (\tilde{e}/\tilde{e}_0) \mathbf{p} f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.34}$$

Using (43.10) and (43.11) we obtain

$$MD = \gamma \int [1 + (\mathbf{v} \cdot \mathbf{p}_0)/\tilde{e}_0][\mathbf{p}_0 + (\gamma - 1)v^{-2}(\mathbf{v} \cdot \mathbf{p}_0)\mathbf{v} + (\gamma \tilde{e}_0/c^2)\mathbf{v}]f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.35}$$

Expanding the integrand, discarding those terms that integrate to zero for isotropic f_0, and using (43.30) we find

$$MD = (\gamma/c^2) \int [\gamma \tilde{e}_0 \mathbf{v} + (\gamma - 1)v^{-2}(\mathbf{U}_0 \cdot \mathbf{v})(\mathbf{p}_0 \cdot \mathbf{v})\mathbf{v} + (\mathbf{U}_0 \cdot \mathbf{v})\mathbf{p}_0]f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.36}$$

Again using (43.21) and (43.24) with $\mathbf{l} = (\mathbf{v}/v)$, we obtain

$$MD = \gamma^2 \rho_{00}\mathbf{v} + \gamma(\gamma - 1)(p_m/c^2)\mathbf{v} + (\gamma/c^2) \int (\mathbf{U}_0 \cdot \mathbf{v})\mathbf{p}_0 f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.37}$$

By resolving $\mathbf{p}_0$ along $\mathbf{l} = (\mathbf{v}/v)$ and a unit vector $\mathbf{m}$ orthogonal to $\mathbf{l}$, it is easy to show that, for an isotropic f_0,

$$\int (\mathbf{U}_0 \cdot \mathbf{v})\mathbf{p}_0 f_0(\mathbf{p}_0)\, d\mathbf{p}_0 \equiv \mathbf{v} \int (\mathbf{U}_0 \cdot \mathbf{l})(\mathbf{p}_0 \cdot \mathbf{l}) f_0(\mathbf{p}_0)\, d\mathbf{p}_0, \tag{43.38}$$

which is just $p_m\mathbf{v}$. Thus (43.37) reduces to

$$MD = \gamma^2(\rho_{00} + p_m/c^2)\mathbf{v} = \gamma^2 \rho_{000}\mathbf{v}, \tag{43.39}$$

which is $(1/c)$ times M^{i0} in the stress-energy tensor, consistent with our earlier interpretation of those components.

THE LABORATORY-FRAME PARTICLE FLUX, ENERGY FLUX, AND MOMENTUM FLUX

The *particle flux density vector* (PF) in the laboratory frame is

$$PF = \int \mathbf{v} f(\mathbf{p})\, d\mathbf{p}. \tag{43.40}$$

Again using (43.17), (43.19), and (43.30) restated in the lab frame (i.e., $\tilde{e}\mathbf{v} = c^2\mathbf{p}$), we find

$$PF = c^2 \int (\mathbf{p}/\tilde{e}_0) f_0(\mathbf{p}_0)\, d\mathbf{p}_0. \tag{43.41}$$

Using (43.11) for $\mathbf{p}$, and retaining only the terms that survive the integration we have

$$PF = \gamma \mathbf{v} \int f_0(\mathbf{p}_0)\, d\mathbf{p}_0 = \gamma N_0 \mathbf{v} = N\mathbf{v}, \tag{43.42}$$

which is identical to the particle flux density vector $N_0 V^\alpha$ introduced in §39.

The *particle energy flux vector* (*EF*) in the laboratory frame is

$$EF = \int \tilde{e}\mathbf{v}f(\mathbf{p})\, d\mathbf{p}. \tag{43.43}$$

Proceeding as above, and using (43.34) and (43.39), this expression reduces to

$$EF = \int (\tilde{e}^2/\tilde{e}_0)\mathbf{v}f_0(\mathbf{p}_0)\, d\mathbf{p}_0 = c^2 \int (\tilde{e}/\tilde{e}_0)\mathbf{p}f_0(\mathbf{p}_0)\, d\mathbf{p}_0 = \gamma^2 \rho_{000} c^2 \mathbf{v}. \tag{43.44}$$

Thus the energy flux is c times the components M^{0i} in the stress-energy tensor, as expected from the interpretation given earlier for those components. It is also equal to c^2 times the momentum density (43.39), consistent with the relationship between M^{0i} and M^{i0}.

Finally, the *particle momentum flux tensor* in the laboratory frame is

$$\Pi^{ij} = \int v^i p^j f(\mathbf{p})\, d\mathbf{p}. \tag{43.45}$$

Again proceeding as above we find

$$\begin{aligned}
\Pi^{ij} &= \int v^i p^j \left(\frac{\tilde{e}}{\tilde{e}_0}\right) f_0(\mathbf{p}_0)\, d\mathbf{p}_0 = c^2 \int \frac{p^i p^j}{\tilde{e}_0} f_0(\mathbf{p}_0)\, d\mathbf{p}_0 \\
&= c^2 \int \frac{1}{\tilde{e}_0}\left[p_0^i + \frac{(\gamma-1)(\mathbf{v}\cdot\mathbf{p}_0)}{v^2} v^i + \frac{\gamma \tilde{e}_0}{c^2} v^i\right] \\
&\quad \times \left[p_o^j + \frac{(\gamma-1)(\mathbf{v}\cdot\mathbf{p}_0)}{v^2} v^j + \frac{\gamma \tilde{e}_0}{c^2} v^j\right] f_0(\mathbf{p}_0)\, d\mathbf{p}_0 \\
&= c^2 \int \frac{1}{\tilde{e}_0}\left[p_0^i p_0^j + \frac{(\gamma-1)^2(\mathbf{v}\cdot\mathbf{p}_0)^2}{v^4} v^i v^j + \frac{\gamma^2 \tilde{e}_0^2}{c^4} v^i v^j \right. \\
&\quad \left. + \frac{(\gamma-1)(\mathbf{v}\cdot\mathbf{p}_0)}{v^2}(p_0^i v^j + p_0^j v^i)\right] f_0(\mathbf{p}_0)\, d\mathbf{p}_0 \\
&= \int \left[U_0^i p_0^j + \frac{(\gamma-1)^2(\mathbf{v}\cdot\mathbf{p}_0)(\mathbf{v}\cdot\mathbf{U}_0)}{v^4} v^i v^j + \frac{\gamma^2 \tilde{e}_0}{c^2} v^i v^j \right. \\
&\quad \left. + \frac{(\gamma-1)(\mathbf{v}\cdot\mathbf{U}_0)}{v^2}(p_0^i v^j + p_0^j v^i)\right] f_0(\mathbf{p}_0)\, d\mathbf{p}_0 \\
&= p_m\left\{\delta^{ij} + [(\gamma-1)^2 + 2(\gamma-1)]\frac{v^i v^j}{v^2}\right\} + \gamma^2 \rho_{00} v^i v^j,
\end{aligned} \tag{43.46}$$

where we have used (43.11), (43.17), (43.21), (43.24), (43.30), (43.38), and the isotropy of f_0. Using the definition of γ to collapse the term in square brackets we obtain finally

$$\Pi^{ij} = p_m\, \delta^{ij} + \gamma^2(\rho_{00} + p_m/c^2) v^i v^j = p_m\, \delta^{ij} + \gamma^2 \rho_{000} v^i v^j, \tag{43.47}$$

which is identical to the space-components M^{ij} in the stress-energy tensor, as expected.

We have thus been able to derive the entire stress-energy tensor from a microscopic point of view.

THE EQUATION OF STATE

Thus far, the only restriction we have placed on $f_0(\mathbf{p}_0)$ is that it be isotropic in the comoving frame. But if the particles move with only nonrelativistic velocities, then it is reasonable to assume that f_0 is the Maxwellian distribution. In the nonrelativistic limit $\tilde{e}_0 = m_0(c^2 + \frac{1}{2}U_0^2)$, hence from (43.21) one easily finds

$$\rho_{00}c^2 = \rho_0 c^2 + \tfrac{3}{2}N_0 kT \tag{43.48}$$

where T is the *material temperature* measured in the comoving frame. Hence the specific internal energy is $e = \frac{3}{2}(N_0kT)/m_0$, and the energy per unit volume is $\hat{e} = \frac{3}{2}N_0kT$. Similarly, from (43.22) one finds

$$-T^{ij} = \rho_0\langle U_0^i U_0^j\rangle = N_0kT\,\delta^{ij}, \tag{43.49}$$

whence $p_m = N_0kT$, and therefore

$$p_m = \tfrac{2}{3}\hat{e} \qquad \text{(N.R.)}. \tag{43.50}$$

All other thermodynamic properties in the comoving frame are the same as those derived in Chapter 1 for a perfect gas.

In the extreme relativistic limit, we see from (37.17) that $\tilde{e}_0 \rightarrow p_0 c$, and from (40.4) that $\rho_{00}c^2 \rightarrow \rho_0 e = \hat{e}$. Thus from (43.21)

$$\hat{e} = c\int p_0 f_0(\mathbf{p}_0)\, d\mathbf{p}_0 = 4\pi c\int p_0 f_0(p_0)p_0^2\, dp_0. \tag{43.51}$$

In this limit we can also write $\mathbf{U}_0 \approx c\mathbf{n}$ where $\mathbf{n} \equiv (\mathbf{p}_0/p_0)$ is the unit vector along $\mathbf{p}_0$. Then, from (43.24), we have

$$p_m = c\int (\mathbf{n}\cdot\mathbf{l})^2 p_0 f_0(\mathbf{p}_0)\, d\mathbf{p}_0 = \tfrac{1}{3}(4\pi c)\int p_0 f_0(p_0)p_0^2\, dp_0 \tag{43.52}$$

because $\langle\mathbf{n}\cdot\mathbf{l}\rangle^2 = \frac{1}{3}$. Thus we obtain the important relation

$$p_m = \tfrac{1}{3}\hat{e}, \qquad \text{(E.R.)} \tag{43.53}$$

which may be contrasted with (43.50) for a nonrelativistic gas. To obtain (43.53) we invoked only the isotropy of f_0. To derive the other thermodynamic quantities for the gas we need an explicit expression for f_0. If the gas is nondegenerate we can use the relativistic generalization of the Maxwellian distribution obtained by eliminating the velocity $\mathbf{v}$ in favor of the momentum $\mathbf{p}$. One then easily can show [cf. (**C1**, Chap. 10)] that $\hat{e} = 3N_0kT$, $p_m = N_0kT$, $c_v = 3R$, $c_p = 4R$, and $\Gamma = \frac{4}{3}$ (here Γ is the adiabatic exponent). Following a different route we shall derive, in §69, the same value for Γ and the same relation between p and $\hat{e}$ for equilibrium radiation (which, of course, is a gas composed of ultrarelativistic particles: photons). Relativistic effects can also be important when the gas is degenerate; see (**C1,** Chap. 24) for details. An in-depth general discussion of relativistic gases is given in (**D1**) and (**S2**).

4.3 Relativistic Dynamics of Nonideal Fluids

Let us now consider a relativistic viscous and heat-conducting fluid viewed as a continuum. We will essentially follow Eckart's ground-breaking analysis (**E1**); for more detailed treatments and discussions of related topics the reader should consult (**L5,** Chap. 5), (**M3,** Chap. 22), (**T1**), (**W1**), and (**W2,** 53–57). Because it causes no additional complication to do so, we assume a general metric tensor $g_{\alpha\beta}$ (with spacelike signature); most of the equations written below will then hold in general relativity as well as special relativity.

44. Kinematics

THE ECKART DECOMPOSITION THEOREM

We define the *projection tensor* to be

$$P^\alpha_\beta \equiv \delta^\alpha_\beta + c^{-2} V^\alpha V_\beta, \tag{44.1}$$

or, in covariant components,

$$P_{\alpha\beta} = g_{\alpha\gamma} P^\gamma_\beta = g_{\alpha\beta} + c^{-2} V_\alpha V_\beta, \tag{44.2}$$

with a similar expression for $P^{\alpha\beta}$. It is easy to prove by direct calculation the following useful relations:

$$P^\alpha_\beta P^\beta_\gamma \equiv P^\alpha_\gamma, \tag{44.3}$$

$$P^{\alpha\beta} \equiv P^{\alpha\gamma} P^\beta_\gamma, \tag{44.4}$$

and

$$P^{\alpha\beta} P_{\alpha\beta} = 3. \tag{44.5}$$

More important, using (36.5), one easily finds that

$$V_\alpha P^\alpha_\beta = V^\beta P^\alpha_\beta \equiv 0, \tag{44.6}$$

which shows that P^α_β produces a projection orthogonal to V^α in spacetime. Recalling that at each point in spacetime V^α lies along the local axis of proper time, we see that P^α_β selects three directions along the local axes of proper space, and thus is the local spatial projection operator in the comoving frame. Indeed, one sees by direct evaluation that

$$(P^\alpha_\beta)_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{44.7}$$

Using P^{α}_{β} we can decompose any vector A^{α} into a scalar a that gives its projection onto the proper time axis, and a vector a^{α} which is the projection of A^{α} into proper space. Thus if we define

$$a \equiv -c^{-2}V_{\alpha}A^{\alpha}, \tag{44.8}$$

and

$$a^{\alpha} \equiv P^{\alpha}_{\beta}A^{\beta}, \tag{44.9}$$

then we can write

$$A^{\alpha} = aV^{\alpha} + a^{\alpha}. \tag{44.10}$$

It is easy to show by direct calculation that (44.8) to (44.10) are mutually consistent.

Similarly, we can decompose any tensor $W^{\alpha\beta}$ into its proper components by defining the scalar

$$w \equiv c^{-4}V_{\alpha}V_{\beta}W^{\alpha\beta}, \tag{44.11}$$

the vector

$$w^{\alpha} \equiv -c^{-2}P^{\alpha}_{\beta}W^{\beta\gamma}V_{\gamma}, \tag{44.12}$$

and the tensor

$$w^{\alpha\beta} \equiv P^{\alpha}_{\gamma}P^{\beta}_{\delta}W^{\gamma\delta}. \tag{44.13}$$

We can then write

$$W^{\alpha\beta} = wV^{\alpha}V^{\beta} + w^{\alpha}V^{\beta} + w^{\beta}V^{\alpha} + w^{\alpha\beta}, \tag{44.14}$$

which is known as the *Eckart decomposition theorem*. Again, it is easy to verify by direct calculation that (44.11) to (44.14) are mutually consistent.

THE VELOCITY-GRADIENT TENSOR

Consider the fluid velocity-gradient tensor $V_{\alpha;\beta}$. Notice first that

$$V_{\alpha;\beta}V^{\beta} \equiv (\delta V_{\alpha}/\delta\tau), \tag{44.15}$$

the intrinsic derivative of V_{α} with respect to proper time; this quantity is the fluid *four-acceleration* A_{α}. We therefore write $V_{\alpha;\beta}$ in terms of an acceleration component along the local time axis, and a set of spatial components, by means of the decomposition

$$V_{\alpha;\beta} = V_{\alpha;\gamma}P^{\gamma}_{\beta} - c^{-2}A_{\alpha}V_{\beta}. \tag{44.16}$$

To verify the correctness of this decomposition we note that

$$V^{\beta}V_{\alpha;\beta} = (V^{\beta}P^{\gamma}_{\beta})V_{\alpha;\gamma} - c^{-2}(V^{\beta}V_{\beta})A_{\alpha} = 0 + A_{\alpha}, \tag{44.17}$$

and

$$P^{\gamma}_{\beta}V_{\alpha;\gamma} = (P^{\gamma}_{\beta}P^{\delta}_{\gamma})V_{\alpha;\delta} - c^{-2}(P^{\gamma}_{\beta}V_{\gamma})A_{\alpha} = P^{\delta}_{\beta}V_{\alpha;\delta} + 0. \tag{44.18}$$

We can decompose the right-hand side of (44.16) further by defining the antisymmetric *rotation tensor*

$$\Omega_{\alpha\beta} \equiv \tfrac{1}{2}(V_{\alpha;\gamma}P^{\gamma}_{\beta} - V_{\beta;\gamma}P^{\gamma}_{\alpha}), \tag{44.19}$$

and the symmetric *shear tensor* (or *rate of strain tensor*)

$$E_{\alpha\beta} \equiv \tfrac{1}{2}(V_{\alpha;\gamma}P^{\gamma}_{\beta} + V_{\beta;\gamma}P^{\gamma}_{\alpha}), \tag{44.20}$$

which are given these names because in the comoving frame they reduce to the covariant generalizations of the Newtonian expressions for these quantities (cf. §21). Clearly $E_{\alpha\beta} + \Omega_{\alpha\beta} \equiv V_{\alpha;\gamma}P^{\gamma}_{\beta}$, hence we can rewrite (44.16) as

$$V_{\alpha;\beta} = E_{\alpha\beta} + \Omega_{\alpha\beta} - c^{-2}A_{\alpha}V_{\beta}. \tag{44.21}$$

It is actually more convenient to choose the shear tensor to be

$$D_{\alpha\beta} \equiv E_{\alpha\beta} - \tfrac{1}{3}\theta P_{\alpha\beta}, \tag{44.22}$$

where

$$\theta \equiv V^{\alpha}_{;\alpha} \tag{44.23}$$

is the *expansion* of the fluid. One sees that $D_{\alpha\beta}$ is the covariant generalization of the Newtonian traceless shear tensor [cf. (25.3) and (32.34)].

Thus we can write, finally,

$$V_{\alpha;\beta} = D_{\alpha\beta} + \Omega_{\alpha\beta} + \tfrac{1}{3}\theta P_{\alpha\beta} - c^{-2}A_{\alpha}V_{\beta}, \tag{44.24}$$

which is the *relativistic generalization of the Cauchy-Stokes theorem* discussed in §21. It shows that in spacetime a fluid is accelerated along its proper time axis, and experiences shear, rotation, and expansion along its local space axes. Explicit expressions for P^{α}_{β}, A_{α}, θ, $\Omega_{\alpha\beta}$, and $D_{\alpha\beta}$ in terms of the ordinary velocity $\mathbf{v}$ and lab-frame space and time derivatives are easily derived. These expressions are too lengthy to reproduce here, but are given by Greenberg in (**G1,** 764–765); the reader should note that the signs in some of Greenberg's formulae conflict with those in our formulae because Greenberg uses a metric with a timelike signature.

Finally, note in passing that from (44.6),

$$V^{\alpha}D_{\alpha\beta} = V_{\alpha}D^{\alpha\beta} \equiv 0, \tag{44.25}$$

and

$$V^{\alpha}\Omega_{\alpha\beta} = V_{\alpha}\Omega^{\alpha\beta} \equiv 0. \tag{44.26}$$

Furthermore, by virtue of (44.4) and (44.5) we have

$$\begin{aligned} P_{\alpha\beta}D^{\alpha\beta} = P^{\alpha\beta}D_{\alpha\beta} &= \tfrac{1}{2}(V_{\alpha;\gamma}P^{\alpha\gamma} - V_{\beta;\gamma}P^{\beta\gamma}) - \theta \\ &= V_{\alpha;\gamma}(g^{\alpha\gamma} + c^{-2}V^{\alpha}V^{\gamma}) - \theta = V^{\gamma}_{;\gamma} - \theta \equiv 0, \end{aligned} \tag{44.27}$$

where we used (36.8) and (44.23). We will find all of these results useful in §§46 and 47.

45. The Stress-Energy Tensor

Given the results of §44 it is fairly straightforward to deduce the form of the stress-energy tensor for viscous and heat-conducting fluids by seeking suitable covariant generalizations of the usual Newtonian expressions. Thus the viscous terms in the stress-energy tensor must be

$$-\mathsf{M}_{\text{viscous}} = 2\mu\,\mathsf{D} + \zeta\theta\mathsf{P}, \tag{45.1}$$

where D is the shear tensor (44.22), P is the projection tensor (44.2), and μ and ζ are, respectively, the coefficients of shear and bulk viscosity. One can see by inspection that, in the comoving frame, (45.1) reduces to the covariant generalization of the Newtonian expression (25.3). (The minus sign appears because of the sign convention for the Cauchy stress tensor; cf. §22.)

Consider now the contribution from heat flow. Classically we describe heat flow in the comoving frame by a vector $\mathbf{q}$, whose components give the rate of energy flow per unit area along each coordinate axis. We saw in §§40 and 43 that the energy flux is c times the elements M^{0i} of the stress-energy tensor. Therefore if we take $\mathbf{Q}$ to be the four-vector generalization of $\mathbf{q}$, then in the comoving frame we must have $(M^{0i})_0 = Q_0^i/c$; we can obtain precisely such a contribution to M_0 from a term of the form $c^{-2}V^\alpha Q^\beta$. But if we introduce such a term, then because M must be symmetric, and from the Eckart decomposition theorem, we know that M must also contain a term of the form $c^{-2}Q^\alpha V^\beta$. We thus conclude that

$$M^{\alpha\beta}_{\text{heat}} = c^{-2}(V^\alpha Q^\beta + Q^\alpha V^\beta). \tag{45.2}$$

As before, the $(i, 0)$ elements of M_{heat} can be interpreted as c times a momentum density. That such terms should be present is reasonable because heat is energy and, by Einstein's mass-energy equivalence, has inertia; hence in the comoving frame a heat flux q^i gives rise to an equivalent mass flux or a momentum density equal to $c^{-2}q^i$, which is, in fact, identical to $(M^{i0}_{\text{heat}})_0/c$.

Adding the viscous and heat-flow contributions to the stress-energy tensor (40.10) for an ideal gas, we conclude that the complete material stress-energy tensor for a viscous, heat-conducting fluid is

$$M_{\alpha\beta} = \rho_{000}V_\alpha V_\beta + pg_{\alpha\beta} - 2\mu D_{\alpha\beta} - \zeta\theta P_{\alpha\beta} + c^{-2}(Q_\alpha V_\beta + V_\alpha Q_\beta), \tag{45.3}$$

or

$$M_{\alpha\beta} = \rho_{00}V_\alpha V_\beta + pP_{\alpha\beta} - 2\mu D_{\alpha\beta} - \zeta\theta P_{\alpha\beta} + c^{-2}(Q_\alpha V_\beta + V_\alpha Q_\beta). \tag{45.4}$$

Notice that a one-to-one correspondence can be made between the terms in (45.4) and those appearing in the Eckart decomposition theorem (44.14). In particular, we can identify Q^α with w^α as given by (44.12); then in view of (44.6) we see that

$$V_\alpha Q^\alpha \equiv 0, \tag{45.5}$$

a result we will find useful below.

Explicit expressions for $M^{\alpha\beta}$ in terms of the ordinary velocity, and lab-frame space and time derivatives, are given in (**G1,** 769–770). To translate the symbols used in (**G1**) to those used here, make the substitutions $\nu \rightarrow 2\mu$, $\beta \rightarrow 3\zeta$, and $\lambda \rightarrow K$ (the thermal conductivity, see §46).

46. *The Energy Equation*

We are now in a position to derive the equations of hydrodynamics for a relativistic nonideal fluid. As for an ideal fluid, the general equations of motion follow from

$$M^{\alpha\beta}_{;\beta} = F^\alpha. \tag{46.1}$$

For simplicity we assume that both μ and ζ are constants. Differentiating each term in (45.4) we have

$$(\rho_{00} V^\alpha V^\beta)_{;\beta} = [(D\rho_{00}/D\tau) + \rho_{00}\theta]V^\alpha + \rho_{00}A^\alpha, \tag{46.2}$$

$$(pP^{\alpha\beta})_{;\beta} = p_{,\beta}P^{\alpha\beta} + pP^{\alpha\beta}_{;\beta}, \tag{46.3}$$

$$\zeta(\theta P^{\alpha\beta})_{;\beta} = \zeta\theta_{,\beta}P^{\alpha\beta} + \zeta\theta P^{\alpha\beta}_{;\beta}, \tag{46.4}$$

$$(Q^\alpha V^\beta)_{;\beta} = \theta Q^\alpha + (DQ^\alpha/D\tau), \tag{46.5}$$

and

$$(V^\alpha Q^\beta)_{;\beta} = V^\alpha Q^\beta_{;\beta} + Q^\beta g^{\alpha\gamma} V_{\gamma;\beta} = V^\alpha Q^\beta_{;\beta} + Q_\beta(D^{\alpha\beta} + \Omega^{\alpha\beta} + \tfrac{1}{3}\theta P^{\alpha\beta}). \tag{46.6}$$

Then, collecting terms and using the relation

$$P^{\alpha\beta}_{;\beta} = (g^{\alpha\beta} + c^{-2}V^\alpha V^\beta)_{;\beta} = c^{-2}(A^\alpha + \theta V^\alpha), \tag{46.7}$$

we have

$$\begin{aligned} M^{\alpha\beta}_{;\beta} = {} & \{(D\rho_{00}/D\tau) + \theta[\rho_{00} + c^{-2}(p - \zeta\theta)]\}V^\alpha \\ & + [\rho_{00} + c^{-2}(p - \zeta\theta)]A^\alpha + P^{\alpha\beta}(p - \zeta\theta)_{,\beta} - 2\mu D^{\alpha\beta}_{;\beta} \\ & + c^{-2}[(DQ^\alpha/D\tau) + \tfrac{4}{3}\theta Q^\alpha + V^\alpha Q^\beta_{;\beta} + Q_\beta(D^{\alpha\beta} + \Omega^{\alpha\beta})] = F^\alpha. \end{aligned} \tag{46.8}$$

We obtain the energy equation by taking the "time" component of (46.8), that is, by projecting it onto V_α. Thus forming $V_\alpha M^{\alpha\beta}_{;\beta}$ and using (36.5), (36.8), (44.6), (44.25), (44.26), and (45.5) we find

$$c^2(D\rho_{00}/D\tau) + (\rho_{00}c^2 + p)\theta = -2\mu V_\alpha D^{\alpha\beta}_{;\beta} + \zeta\theta^2 - [Q^\beta_{;\beta} - c^{-2}V_\alpha(DQ^\alpha/D\tau)], \tag{46.9}$$

where, as in §42, we have assumed that the only forces acting are such that $V_\alpha F^\alpha \equiv 0$.

We can rewrite the left-hand side of (46.9) in a more useful form by noticing that

$$\begin{aligned} V^\alpha(\rho_{00})_{,\alpha} + \rho_{000}V^\alpha_{;\alpha} &= V^\alpha(\rho_{000})_{,\alpha} + \rho_{000}V^\alpha_{;\alpha} - c^{-2}V^\alpha p_{,\alpha} \\ &= (\rho_{000}V^\alpha)_{;\alpha} - c^{-2}V^\alpha p_{,\alpha}. \end{aligned} \tag{46.10}$$

Similarly, on the right-hand side, using (45.5) we have

$$V_\alpha(DQ^\alpha/D\tau) = [D(V_\alpha Q^\alpha)/D\tau] - Q^\alpha A_\alpha \equiv -Q^\alpha A_\alpha, \qquad (46.11)$$

and using (44.25) we have

$$V_\alpha D^{\alpha\beta}_{;\beta} = (V_\alpha D^{\alpha\beta})_{;\beta} - D^{\alpha\beta} V_{\alpha;\beta} \equiv -D^{\alpha\beta} V_{\alpha;\beta}. \qquad (46.12)$$

Then substituting for $V_{\alpha;\beta}$ from (44.24), and using (44.25) and (44.27) we see that

$$D^{\alpha\beta} V_{\alpha;\beta} \equiv D^{\alpha\beta} D_{\alpha\beta}. \qquad (46.13)$$

Using (46.10) to (46.13) in (46.9) we find

$$(\rho_{000} c^2 V^\alpha)_{;\alpha} - V^\alpha p_{,\alpha} = 2\mu D^{\alpha\beta} D_{\alpha\beta} + \zeta\theta^2 - (Q^\beta_{;\beta} + c^{-2} Q^\alpha A_\alpha), \qquad (46.14)$$

Then by the same steps that lead from (42.8) to (42.10) we can reduce (46.14) to

$$\rho_0 T \frac{Ds}{D\tau} \equiv \rho_0 \left[\frac{De}{D\tau} + p \frac{D}{D\tau}\left(\frac{1}{\rho_0}\right)\right] = 2\mu D^{\alpha\beta} D_{\alpha\beta} + \zeta\theta^2 - \left(Q^\beta_{;\beta} + \frac{Q^\alpha}{c^2} A_\alpha\right), \qquad (46.15)$$

where s is the specific entropy.

Equation (46.15) is the relativistic generalization of the entropy generation equation (27.11) when we use (27.28) for the dissipation function Φ. As in the nonrelativistic limit, the first two terms on the right-hand side correspond to irreversible heat generation by viscous dissipation. The third term gives the rate of heat flow into the material from its surroundings. The fourth term is purely relativistic in origin and predicts an additional deposition of heat when the material accelerates into the heat flow (**a** and **q** antiparallel), which is reasonable because if we suppose that the heat flux arises from radiation, then we see that more heat can be delivered to material accelerating into the radiation flow because photons will be blueshifted to higher energies as they enter the fluid element and redshifted to lower energies as they leave it.

Thus far we have left the form of Q^α unspecified, although we expect it to reduce to Fourier's law $\mathbf{q} = -K\,\nabla T$ in the nonrelativistic limit. We can deduce an expression for Q^α as follows. From (27.19), we know that classically we must have

$$\int_{\mathcal{V}} \rho \frac{Ds}{Dt}\, dV + \int_S \frac{\mathbf{q}\cdot\mathbf{n}}{T}\, dS = \int_{\mathcal{V}} \left[\frac{\partial}{\partial t}(\rho s) + \nabla \cdot \left(\rho s \mathbf{v} + \frac{\mathbf{q}}{T}\right)\right] dV \geq 0, \qquad (46.16)$$

where we used (19.3) and the divergence theorem. Because $\mathcal{V}$ is arbitrary,

$$(\rho s)_{,t} + \nabla \cdot [\rho s \mathbf{v} + (\mathbf{q}/T)] \geq 0. \qquad (46.17)$$

From (46.17) we see that the vector $\mathbf{s} = \rho s \mathbf{v} + (\mathbf{q}/T)$ can be interpreted classically as the entropy flux density in a heat-conducting fluid [in an ideal fluid the term $\mathbf{q}/T$ is, of course, absent; cf. (27.14)]. The covariant generalization of $\mathbf{s}$ is

$$S^\alpha = \rho_0 s V^\alpha + (Q^\alpha/T), \tag{46.18}$$

which we take to be the *entropy flux density four-vector.* As the covariant generalization of (46.17) we therefore take

$$S^\alpha_{;\alpha} \geq 0, \tag{46.19}$$

which is, in essence, a relativistic statement of the second law of thermodynamics.

Substituting (46.18) into (46.19), and using (39.10), we have

$$\rho_0(Ds/D\tau) + (Q^\alpha/T)_{;\alpha} \geq 0. \tag{46.20}$$

Combining (46.20) and (46.15) we thus find

$$\rho_0(Ds/D\tau) + (Q^\alpha/T)_{;\alpha} = (2\mu D^{\alpha\beta}D_{\alpha\beta} + \zeta\theta^2)/T - (Q^\alpha/T^2)(T_{,\alpha} + c^{-2}TA_\alpha) \geq 0. \tag{46.21}$$

The first term on the right-hand side is obviously positive, so we must merely choose Q^α in such a way as to guarantee that the second term will be positive. Eckart (**E1**) noted that the simplest way to do this is to take

$$Q^\alpha = -KP^{\alpha\beta}(T_{,\beta} + c^{-2}TA_\beta), \tag{46.22}$$

which (1) is consistent with Fourier's law in the classical limit, (2) is consistent with the requirements of the Eckart decomposition theorem [cf. (44.12) and (45.5)], and (3) makes the second term on the right-hand side a positive perfect square, guaranteeing positivity, as desired. The term TA_β is relativistic in origin and implies a flow of heat in accelerated matter even if the material is isothermal; the flow is in the direction opposite to the acceleration, and can be ascribed to the inertia of the heat energy [see (**E1**) for further discussion and interpretation]. Finally, using (46.22) in (46.21), and evaluating the result in the comoving frame we find

$$\rho_0\left(\frac{Ds}{D\tau}\right) = \frac{K}{T}\nabla\cdot\left(\nabla T + \frac{T\mathbf{a}_0}{c^2}\right) + \frac{K}{T^2}\left(\nabla T + \frac{T\mathbf{a}_0}{c^2}\right)^2 + \frac{\Phi_0}{T}, \tag{46.23}$$

which is a direct analogue of the classical result (27.19) when the terms containing $\mathbf{a}_0$ are suppressed.

Explicit expressions for Q^α and for the energy equation (46.15) in terms of the ordinary velocity, and lab-frame space and time derivatives, are given in (**G1,** 765–766); again conflicts of signs arise in the formulae because of Greenberg's choice of a timelike signature.

47. *The Equations of Motion*

To obtain the equations of motion for a relativistic nonideal fluid we take the "space" components of (46.8) by calculating $P_{\alpha\gamma}M^{\gamma\beta}_{;\beta}$. Noting that

$$P_{\alpha\gamma}A^\gamma = A_\alpha \tag{47.1}$$

and

$$P_{\alpha\gamma}F^{\gamma} = F_{\alpha}, \tag{47.2}$$

we easily find

$$\begin{aligned}[\rho_{00} + c^{-2}(p - \zeta\theta)](DV_{\alpha}/D\tau) &= F_{\alpha} - P_{\alpha}^{\beta}(p - \zeta\theta)_{,\beta} + 2\mu P_{\alpha\gamma}D^{\gamma\beta}_{;\beta} \\ &\quad - c^{-2}[P_{\alpha\gamma}(DQ^{\gamma}/D\tau) + \tfrac{4}{3}\theta Q_{\alpha} + P_{\alpha\gamma}Q_{\beta}(D^{\gamma\beta} + \Omega^{\gamma\beta})].\end{aligned} \tag{47.3}$$

To obtain the nonrelativistic limit we let $c \to \infty$. Then in Cartesian coordinates $V_i \to v_i$, $(D/D\tau) \to (D/Dt)$, $P_{ij} \to \delta_{ij}$, $P^i_j \to \delta^i_j$, and we recover the usual Navier-Stokes equation (26.1), as expected.

Explicit expressions for the momentum equations in terms of the ordinary velocity, and lab-frame space and time derivatives, are given in (**G1,** 767–768). As before, it is necessary to translate some of the notation.

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